1) $x' = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{A} x \qquad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\det(A - rI) = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \implies$ Eigenvalues 3 and -1

For $\lambda = 3$:

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies 2v_1 = v_2$$

so a eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ one solution of the d.e. is $\begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = v^{(1)}$

For $\lambda = -1$:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -2v_1 = v_2$$

eigenvector : $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

solution of d.e. : $\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = v^{(2)}$

General solution : $x = c_1 v^{(1)} + c_2 v^{(2)} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{bmatrix}$

$$x(0) = \begin{bmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \implies \begin{cases} c_1 = \frac{7}{4} \\ c_2 = +\frac{1}{4} \end{cases}$$

So $\boxed{x = \begin{bmatrix} \frac{7}{4}e^{3t} + \frac{1}{4}e^{-t} \\ \frac{7}{2}e^{3t} - \frac{1}{2}e^{-t} \end{bmatrix}}$ is the solution of the IVP.

2) $x' = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{aligned} \det(A-\lambda I) &= (3-\lambda)\left((3-\lambda)(-1-\lambda)+3\right) + 1 + \lambda - 3 - 3 + 9 - 3\lambda \\ &= (3-\lambda)(\cancel{-3} - 2\lambda + \lambda^2 + \cancel{3}) - 2\lambda + 4 \\ &= (3-\lambda)\lambda(\lambda-2) - 2(\lambda-2) \\ &= \left((3-\lambda)\lambda - 2\right)(\lambda-2) \\ &= (-\lambda^2 + 3\lambda - 2)(\lambda-2) \\ &= -(\lambda-1)(\lambda-2)(\lambda-2) \end{aligned}$$

(could also have factored this by first guessing one of the roots, of course.)

Eigenvalues: 1, 2

Eigenvectors:

$\lambda = 1$: $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ 3 & 3 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -3 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So, a eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$\lambda = 2$: Solving the linear system in this case gives two linearly independent eigenvectors, since

$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \iff v_1 + v_2 = v_3$.

(only one equ., so two free parameters)

There are lots of solutions of $v_1 + v_2 = v_3$, but two possible linearly independent ones are:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So, general solution of the d.e. is

$$x = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

To satisfy the initial conditions, we need

$$x(0) = \begin{bmatrix} c_1 + c_2 & \\ c_1 & + c_3 \\ 3c_1 + c_2 & + c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -2 \\ 3 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right)$$

$$\Rightarrow \; c_3 = -2$$
$$c_2 = 3 + c_3 = 1$$
$$c_1 = -2 - c_3 = 0$$

So $$x = \begin{bmatrix} e^{2t} & \\ 0 & -2e^{2t} \\ e^{2t} & -2e^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ -2e^{2t} \\ -e^{2t} \end{bmatrix}$$

6) $X' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} X$

$$\det \begin{pmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{pmatrix} = -(3+\lambda)(1+\lambda)\lambda + 2(-1-2(1+\lambda))$$

$$= -(3+\lambda)(1+\lambda)\lambda - 2 - 4 - 4\lambda$$

$$= -(3+\lambda)(1+\lambda)\lambda - 4\lambda - 6$$

$$= -\lambda^3 - 4\lambda^2 - 3\lambda - 4\lambda - 6$$

So, we need $\quad \lambda^3 + 4\lambda^2 + 7\lambda + 6 = 0$

Guess one root: $\lambda = -2$: $\quad -8 + 4(+4) + 7(-2) + 6$

$$= -8 + 16 - 14 + 6$$

$$= 0 \checkmark$$

So divide out $\lambda + 2$:

$$\begin{array}{r} \lambda^2 + 2\lambda + 3 \\ \lambda + 2 \overline{\big) \lambda^3 + 4\lambda^2 + 7\lambda + 6} \\ \underline{\lambda^3 + 2\lambda^2} \qquad\qquad \\ 2\lambda^2 + 7\lambda \qquad \\ \underline{2\lambda^2 + 4\lambda} \qquad \\ 3\lambda + 6 \\ \underline{3\lambda + 6} \\ 0 \end{array}$$

So $\quad \lambda^3 + 4\lambda^2 + 7\lambda + 6 = (\lambda + 2)(\lambda^2 + 2\lambda + 3)$

So, either $\lambda = -2$ or $\lambda = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm \sqrt{2}\, i$

Eigenvectors:

$\lambda = -2$:
$$\left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$
add $1\times$ row 1 to row 2
& add $(-2)\times$ row 1 to row 3

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \iff \begin{array}{l} v_1 = 2v_3 \\ v_2 = -2v_3 \\ v_3 \text{ arbitrary} \end{array}$$

eigenvector: $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\lambda = -1+\sqrt{2}\,i$:
$$\left(\begin{array}{ccc|c} -3-(-1+\sqrt{2}i) & 0 & 2 & 0 \\ 1 & -1-(-1+\sqrt{2}i) & 0 & 0 \\ -2 & -1 & -(-1+\sqrt{2}i) & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -2-\sqrt{2}i & 0 & 2 & 0 \\ 1 & -\sqrt{2}i & 0 & 0 \\ -2 & -1 & 1-\sqrt{2}i & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 6 & 0 & 2(-2+\sqrt{2}i) & 0 \\ 1 & -\sqrt{2}i & 0 & 0 \\ 0 & -1-2\sqrt{2}i & 1-\sqrt{2}i & 0 \end{array}\right)$$
mult. row 1 by $(-2+\sqrt{2}i)$
& add $2\times$ row 2 to row 3

$$\sim \left(\begin{array}{ccc|c} 0 & 6\sqrt{2}i & -4+2\sqrt{2}i & 0 \\ 1 & -\sqrt{2}i & 0 & 0 \\ 0 & 9 & 3+3\sqrt{2}i & 0 \end{array}\right)$$
add $(-6)\times$ row 2 to row 1
& mult. row 3 by $-1+2\sqrt{2}i$

$$\sim \left(\begin{array}{ccc|c} 0 & 12 & 4\sqrt{2}i+4 & 0 \\ 1 & -\sqrt{2}i & 0 & 0 \\ 0 & 9 & 3+3\sqrt{2}i & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -\sqrt{2}i & 0 & 0 \\ 0 & 3 & 1+\sqrt{2}i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\iff \begin{cases} v_1 = \sqrt{2}i\,v_2 \\ 3v_2 + (1+\sqrt{2}i)v_3 = 0 \end{cases}$$

If we pick $v_3 = 3$, we get

$v_2 = -(1+\sqrt{2}i)$

$v_1 = \sqrt{2}i\,v_2 = -\sqrt{2}i + 2$

so an eigenvector is

$$\begin{bmatrix} 2-\sqrt{2}i \\ -1-\sqrt{2}i \\ 3 \end{bmatrix}$$

A complex solution is

$$e^{(-1+\sqrt{2}i)t}\begin{bmatrix} 2-\sqrt{2}i \\ -1-\sqrt{2}i \\ 3 \end{bmatrix} = e^{-t}(\cos\sqrt{2}t + i\sin\sqrt{2}t)\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + i\begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}\right)$$

So, two real-valued solutions are the real and imaginary parts of this. Namely,

$$e^{-t}\left(\cos\sqrt{2}t\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \sin\sqrt{2}t\begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}\right) = X^{(1)}$$

and $$e^{-t}\left(\cos\sqrt{2}t\begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix} + \sin\sqrt{2}t\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = X^{(2)}$$

are real solutions. The solution from the eigenvalue $\lambda = -2$ is

$$e^{-2t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = X^{(3)}$$

The general solution is any linear combination of these 3 solutions:

$$X = c_1X^{(1)} + c_2X^{(2)} + c_3X^{(3)}$$

$$= e^{-t}\left(\cos\sqrt{2}t\begin{bmatrix} 2c_1-\sqrt{2}c_2 \\ -c_1-\sqrt{2}c_2 \\ 3c_1 \end{bmatrix} + \sin\sqrt{2}t\begin{bmatrix} 2c_2+\sqrt{2}c_1 \\ -c_2+\sqrt{2}c_1 \\ 3c_2 \end{bmatrix}\right) + e^{-2t}\begin{bmatrix} 2c_3 \\ -2c_3 \\ c_3 \end{bmatrix}$$

To satisfy the initial conditions, we need

$$X(0) = \begin{bmatrix} 2c_1-\sqrt{2}c_2+2c_3 \\ -c_1-\sqrt{2}c_2-2c_3 \\ 3c_1+c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}.$$

You can check that this linear system has solution $c_1 = -1$, $c_2 = 0$, $c_3 = 1$

So

$$\boxed{X = e^{-t}\left(\cos\sqrt{2}t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + \sin\sqrt{2}t\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}\right) + e^{-2t}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$$