MAT 22B

HW #8 Solutions

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1) $x' = \underbrace{\begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}}_{\substack{\shortparallel \\ A}} x$

Char. poly.: $\det(A-\lambda I) =$

$$(1-\lambda)((1-\lambda)(4-\lambda)+2) - 2((4-\lambda)-2) - 3(-1-(1-\lambda))$$
$$= (1-\lambda)(6-5\lambda+\lambda^2) - 4 + 2\lambda + 6 - 3\lambda$$
$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3$$
$$= (2-\lambda)^3$$

So only eigenvalue is 2.

Eigenvectors:

$$(A-2I)v = \begin{pmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

solve by row reduction:

$$\left(\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $v_2 - v_3 = 0$, $v_1 + v_3 = 0$

Picking $v_3 = 1$ we get $v_2 = 1$, $v_1 = -1$ so $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

The corresponding solution of the diff. eq. is $e^{At}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \boxed{e^{2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$.

We need two more solutions, so we look for generalized eigenvectors:

$(A-2I)^2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ so we want v s.t. $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

The augmented matrix is row-equivalent to $\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ so

we get only one equation: $v_2 - v_3 = 0$. This can be satisfied in many ways, e.g. $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, but we can't use both of these, since we need a set of three vectors (including the eigenvector $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$) that are <u>linearly independent</u>. Let's choose $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The solution coming from this generalized eigenvector is

$$e^{At}\begin{pmatrix}1\\0\\0\end{pmatrix} = e^{2t}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + t\underbrace{\begin{pmatrix}-1 & 2 & -3\\ 1 & -1 & 2\\ 1 & -1 & 2\end{pmatrix}}_{A-2I}\begin{pmatrix}1\\0\\0\end{pmatrix}\right]$$

$$= e^{2t}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\1\\1\end{pmatrix}\right] = \boxed{e^{2t}\begin{pmatrix}1-t\\t\\t\end{pmatrix}}$$

We still need one more solution, so look for vectors v such that $(A-\lambda I)^3 v = 0$:

$$(A-2I)^3 = (A-2I)^2(A-2I) = \begin{pmatrix}0 & -1 & 1\\ 0 & 1 & -1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}-1 & 2 & -3\\ 1 & -1 & 2\\ 1 & -1 & 2\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

So we need vectors v satisfying

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$ Any vector v satisfies this. We just need to pick one that is linearly independent of our previous vectors $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$. Let's pick $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

(to verify that these are linearly independent we could calculate $\det\begin{pmatrix}-1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$)

So, our final solution is

$$e^{At}\begin{pmatrix}0\\1\\0\end{pmatrix} = e^{2t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\underbrace{\begin{pmatrix}-1 & 2 & -3\\ 1 & -1 & 2\\ 1 & -1 & 2\end{pmatrix}}_{A-2I}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{2}t^2\underbrace{\begin{pmatrix}0 & -1 & 1\\ 0 & 1 & -1\\ 0 & 1 & -1\end{pmatrix}}_{(A-2I)^2}\begin{pmatrix}0\\1\\0\end{pmatrix}\right]$$

$$= e^{2t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\-1\\-1\end{pmatrix} + \frac{1}{2}t^2\begin{pmatrix}-1\\1\\1\end{pmatrix}\right] = \boxed{e^{2t}\begin{pmatrix}2t-\frac{1}{2}t^2\\ 1-t+\frac{1}{2}t^2\\ -t+\frac{1}{2}t^2\end{pmatrix}}$$

So, we have found three linearly independent solutions, as we wished to.

(Incidentally, this means a fundamental matrix solution is $e^{2t}\begin{pmatrix}-1 & 1-t & 2t-\frac{1}{2}t^2\\ +1 & t & 1-t+\frac{1}{2}t^2\\ 1 & t & -t+\frac{1}{2}t^2\end{pmatrix}$.)

2) $$X' = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}^{A} X \qquad X(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Char. poly: $\det \begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda(\lambda^2+1) - (-\lambda) = -\lambda^3$

Eigenvalue: $\lambda = 0$ (multiplicity 3).

$A - \lambda I = A$, so we need vectors v s.t. $Av = 0$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so eigenvectors satisfy
$V_1 - V_3 = 0$
$V_2 = 0$

$\Rightarrow$ only get one linearly indep. eigenvector, say $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Find generalized eigenvectors:

$$(A - \lambda I)^2 = A^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad V_1 - V_3 = 0.$$

Only one equation, so we have two free variables, <u>but</u> we need vectors linearly independent of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, so we only get one vector, say $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. (Any other vector satisfying $v_1 - v_3 = 0$ would have to be a linear combination of these two).

Need one more vector, so look for v satisfying $(A - \lambda I)^3 v = 0$.

$$(A - \lambda I)^3 = A^3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\Rightarrow 0v_1 + 0v_2 + 0v_3 = 0$, which is satisfied by <u>any</u> vector; just pick one that is linearly independent of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, say $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Solutions of the diff eq.:

- Since $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is an eigenvector with eigenvalue 0,

$$e^{At}\begin{pmatrix}1\\0\\1\end{pmatrix} = e^{0t}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

- Since $\begin{pmatrix}0\\1\\0\end{pmatrix}$ satisfies $(A-0I)^2v=0$,

$$e^{At}\begin{pmatrix}0\\1\\0\end{pmatrix} = e^{0t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\underbrace{\begin{pmatrix}0&1&0\\1&0&-1\\0&1&0\end{pmatrix}}_{A-0I}\begin{pmatrix}0\\1\\0\end{pmatrix}\right] = \begin{pmatrix}t\\1\\t\end{pmatrix}$$

- Since $\begin{pmatrix}0\\0\\1\end{pmatrix}$ satisfies $(A-0I)^3v=0$,

$$e^{At}\begin{pmatrix}0\\0\\1\end{pmatrix} = e^{0t}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} + t\underbrace{\begin{pmatrix}0&1&0\\1&0&-1\\0&1&0\end{pmatrix}}_{A-0I}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{t^2}{2}\underbrace{\begin{pmatrix}1&0&-1\\0&0&0\\1&0&-1\end{pmatrix}}_{(A-0I)^2}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$= \begin{pmatrix}-\frac{1}{2}t^2\\-t\\-\frac{1}{2}t^2+1\end{pmatrix}$$

So the general solution is any linear combination of these:

$$x = c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}t\\1\\t\end{pmatrix} + c_3\begin{pmatrix}-\frac{1}{2}t^2\\-t\\-\frac{1}{2}t^2+1\end{pmatrix}.$$

To satisfy the initial values, we need

$$x(0) = c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$$

The solution to this system is $c_1 = 1$ $c_2 = -1$ $c_3 = 1$

So

$$\boxed{x = \begin{pmatrix}1-t-\frac{1}{2}t^2\\-1-t\\2-t-\frac{1}{2}t^2\end{pmatrix}}$$

(It's always a good idea to check your answer, of course! Let's check:

$$x' = \begin{pmatrix} -1-t \\ -1 \\ -1-t \end{pmatrix}$$

$$Ax = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1-t-\frac{1}{2}t^2 \\ -1-t \\ 2-t-\frac{1}{2}t^2 \end{pmatrix} = \begin{pmatrix} -1-t \\ -1 \\ -1-t \end{pmatrix} \quad \checkmark$$

So we really have a solution, and

$$x(0) = \begin{pmatrix} 1-0-\frac{1}{2}0^2 \\ -1-0 \\ 2-0-\frac{1}{2}0^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{as we wished.})$$

BTW, another way to solve the initial value problem is to first write down a fundamental matrix solution, and find e^{At}:

$$\Phi = \begin{pmatrix} 1 & t & -\frac{1}{2}t^2 \\ 0 & 1 & -t \\ 1 & t & -\frac{1}{2}t^2+1 \end{pmatrix}$$

so $\Phi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. To calculate $\Phi(0)^{-1}$, we row reduce:

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}}_{\Phi(0)} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) \sim \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}}_{\Phi(0)^{-1}}\right), \quad \text{and then}$$

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} 1 & t & -\frac{1}{2}t^2 \\ 0 & 1 & -t \\ 1 & t & -\frac{1}{2}t^2+1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}t^2 & t & -\frac{1}{2}t^2 \\ t & 1 & -t \\ \frac{1}{2}t^2 & t & -\frac{1}{2}t^2+1 \end{pmatrix}.$$

So:

$$x(t) = e^{At}x(0) = \begin{pmatrix} 1+\frac{1}{2}t^2 & t & -\frac{1}{2}t^2 \\ t & 1 & -t \\ \frac{1}{2}t^2 & t & -\frac{1}{2}t^2+1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-t-\frac{1}{2}t^2 \\ -1-t \\ 2-t-\frac{1}{2}t^2 \end{pmatrix}$$

as we found before!

(Having e^{At} lets us immediately solve any initial value problem by matrix multiplication!)

3) $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$. Find e^{At}.

Eigenvalues: $(-3-\lambda)(-1-\lambda) + 2 = 0$

$$\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

Eigenvectors:

$\lambda = -2+i$: $\left(\begin{array}{cc|c} -3-(-2+i) & 2 & 0 \\ -1 & -1-(-2+i) & 0 \end{array}\right) \sim \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -1+i \\ -1-i & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1+i \\ 0 & 0 \end{pmatrix}$$

so $v_1 = (1-i)v_2$, and an eigenvector is $\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

A complex solution of the diff. eq. $x' = Ax$ is therefore

$$e^{(-2+i)t}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} = e^{-2t}(\cos t + i \sin t)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= \underbrace{e^{-2t}\left(\cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)} + i\, \underbrace{e^{-2t}\left(\sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)}$$

two real-valued solutions.

$$= e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i e^{-2t}\begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$$

A fundamental matrix solution is

$$X = e^{-2t}\begin{pmatrix} \cos t + \sin t & \sin t - \cos t \\ \cos t & \sin t \end{pmatrix}$$

$\mathbb{X}(0) = e^0 \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. To find $\mathbb{X}(0)^{-1}$, use row reduction, or just use the standard way to get the inverse of a 2×2 matrix:

$$\mathbb{X}(0)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

So:

$$e^{At} = \mathbb{X}(t)\mathbb{X}(0)^{-1}$$

$$= e^{-2t}\begin{pmatrix} \cos t + \sin t & \sin t - \cos t \\ \cos t & \sin t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= e^{-2t}\begin{pmatrix} \cos t - \sin t & 2\sin t \\ -\sin t & \cos t + \sin t \end{pmatrix}$$

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4) $x' + Bx = g$. Multiply this equation (on the left) by the matrix e^{Bt}. Then

$$e^{Bt}x' + e^{Bt}Bx = e^{Bt}g$$

For this step, note that

$$e^{Bt}B = (I + Bt + \tfrac{1}{2}B^2t^2 + \tfrac{1}{6}B^3t^3 + \cdots)B$$
$$= (B + B^2t + \tfrac{1}{2}B^3t^2 + \tfrac{1}{6}B^4t^3 + \cdots)$$
$$= B(I + Bt + \tfrac{1}{2}B^2t^2 + \cdots)$$
$$= Be^{Bt}$$

$$e^{Bt}x' + Be^{Bt}x = e^{Bt}g$$

$$\frac{d}{dt}(e^{Bt}x) = e^{Bt}g$$

product rule for matrix-valued functions!

$$e^{Bt}x = \int e^{Bt}g\,dt$$

$$x = (e^{Bt})^{-1}\int e^{Bt}g\,dt$$

$$= e^{-Bt}\int e^{Bt}g\,dt$$

I showed in class why e^{Bt} is invertible and $e^{-Bt} = (e^{Bt})^{-1}$.

5) $x' + \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} x = \begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix}$.

From the previous problem, we know that the solution is

$$x = e^{\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} t} \int e^{\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} t} \begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} dt$$

We calculated in problem 3 that $e^{\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} t} = e^{-2t} \begin{pmatrix} \cos t - \sin t & 2\sin t \\ -\sin t & \cos t + \sin t \end{pmatrix}$,

and therefore, $e^{\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} t} = \left(e^{\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} t} \right)^{-1} = e^{+2t} \begin{pmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{pmatrix}$

So:

$$x = e^{2t} \begin{pmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{pmatrix} \int \begin{pmatrix} \cos t - \sin t & 2\sin t \\ -\sin t & \cos t + \sin t \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} dt$$

$$= e^{2t} \begin{pmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{pmatrix} \int \begin{pmatrix} \cos t + 5\sin t \\ 3\cos t + 2\sin t \end{pmatrix} dt$$

$$= e^{2t} \begin{pmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{pmatrix} \begin{pmatrix} \sin t - 5\cos t + c_1 \\ 3\sin t - 2\cos t + c_2 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} -5\cos^2 t - 5\sin^2 t + 0\cos t\sin t + c_1\cos t + (c_1 - 2c_2)\sin t \\ -2\sin^2 t - 2\cos^2 t + 0\cos t \sin t + (c_1 - c_2)\sin t + c_2\cos t \end{pmatrix}$$

$$x = \boxed{e^{2t} \begin{pmatrix} -5 + c_1\cos t + (c_1 - 2c_2)\sin t \\ -2 + c_2\cos t + (c_1 - c_2)\sin t \end{pmatrix}}$$

general solution, where c_1 & c_2 are arbitrary constants.