Summary of 1st order linear systems $\mathbf{x}' = A\mathbf{x}$

D.K. Wise
MAT-22B

Let A be a constant $n \times n$ matrix, $\mathbf{x} = \mathbf{x}(t)$ an $n \times 1$ matrix (a 'vector') that is a function of t.

- Just as the solution of $x' = ax$ is $e^{at}c$, where $c = x(0)$ is a constant, the solution of $\mathbf{x}' = A\mathbf{x}$ is $e^{At}\mathbf{v}$, where $\mathbf{v}$ is a constant vector. Here, e^{At} is the $n \times n$ matrix defined by the power series:
$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \cdots$$
This can be shown to converge for any value of t. It can be differentiated term by term to prove that $\frac{d}{dt}e^{At} = Ae^{At}$, and hence that $e^{At}\mathbf{v}$ solves the differential equation for any choice of $\mathbf{v}$.

- Calculating e^{At} directly by the power series is hard (except in some very special cases, like when A is a diagonal matrix). So, even though we *know* the solution of $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{v}$ is $\mathbf{x} = e^{At}\mathbf{v}$, to explicitly calculate this solution, we must use some clever trick . . .

- Strategy: instead of trying to calculate e^{At} itself, find n linearly independent vectors $\mathbf{v}$ for which $e^{At}\mathbf{v}$ is a *finite* series. Observe:
$$e^{At}\mathbf{v} = e^{\lambda t} e^{t(A-\lambda I)}\mathbf{v} = e^{\lambda t}\left[\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{1}{2}t^2(A - \lambda I)^2\mathbf{v} + \cdots\right]$$
If λ is an eigenvalue, and $\mathbf{v}$ is an eigenvector, then $(A - \lambda I)\mathbf{v} = (A - \lambda I)^2\mathbf{v} = \cdots = 0$ so all terms after the first one vanish and we just get:
$$e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v}$$
So, if A has n linearly independent eigenvectors $\mathbf{v}^{(1)}, \ldots \mathbf{v}^{(n)}$, with eigenvectors $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), then we have the general solution of $\mathbf{x}' = A\mathbf{x}$:
$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^{(1)} + \cdots + c_1 e^{\lambda_n t}\mathbf{v}^{(n)}$$
However. . .

- If there are repeated eigenvalues, it is possible we won't get n linearly independent eigenvectors. In this case, for each repeated eigenvalue λ, we look for "generalized eigenvectors" $\mathbf{v}$ satisfying
$$(A - \lambda I)^k\mathbf{v} = 0 \qquad \text{for some } k > 1.$$
For such vectors, the series for e^{At} is still finite:
$$e^{At}\mathbf{v} = e^{\lambda t}\left[\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{1}{2}t^2(A - \lambda I)^2\mathbf{v} + \cdots + \frac{1}{(k-1)!}t^{k-1}(A - \lambda I)^{k-1}\mathbf{v}\right]$$
When λ is an eigenvalue of algebraic multiplicity m, there are always m linearly independent vectors that are either eigenvectors or *generalized* eigenvectors. Finding all of these, for each eigenvalue, we get a total of n linearly independent vectors $\mathbf{v}^{(1)}, \ldots \mathbf{v}^{(n)}$ for which we can compute $e^{At}\mathbf{v}^{(i)}$ exactly using the formula above, and we can write the general solution of the differential equation as:
$$\mathbf{x}(t) = c_1 e^{At}\mathbf{v}^{(1)} + \cdots + c_1 e^{At}\mathbf{v}^{(n)}$$

- **Complex solutions.** While everything above is completely general, when the eigenvalues and (generalized) eigenvectors are *complex*, the solutions $e^{At}\mathbf{v}$ are complex-valued; it takes a bit more work to extract *real* solutions. Luckily, we can get *two* linearly independent real solutions by just taking the real and imaginary parts of *one* solution from a complex conjugate pair. For example, suppose $n = 3$ and two of our eigenvalues are $-3 + 2i$ and $-3 - 2i$, with corresponding eigenvectors
$$\begin{bmatrix} 1 \\ 1+i \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1-i \\ 3 \end{bmatrix}$$
Then one complex-valued solution of the differential equation is:
$$e^{(-3+2i)t}\begin{bmatrix} 1 \\ 1+i \\ 3 \end{bmatrix} = e^{-3t}(\cos 2t + i\sin 2t)\left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$
Two *real*-valued solutions are just the real and imaginary parts of this, namely:
$$e^{-3t}\left(\cos 2t\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \sin 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad e^{-3t}\left(\sin 2t\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + \cos 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$
are real-valued solutions. Take linear combinations of these with the solution you get from the *real* eigenvalue, and you have the general solution. (We never needed the second complex eigenvalue or eigenvector!)

- Given n linearly independent solutions $\mathbf{x}^{(1)}, \ldots, x^{(n)}$ of $\mathbf{x}' = A\mathbf{x}$, let X be the $n \times n$ matrix whose ith column is just the vector $\mathbf{x}^{(i)}$. Then X is called a **fundamental matrix solution** of $\mathbf{x}' = A\mathbf{x}$. Some important properties:
  (1) $X' = AX$. (This is why we call it a fundamental matrix *solution*: it satisfies the differential equation, since each of its columns do.)
  (2) e^{At} is in fact a fundamental matrix solution—its columns are solutions of $\mathbf{x}' = A\mathbf{x}$.
  (3) Given *any* fundamental matrix solution X, we can calculate e^{At} using the formula:
$$e^{At} = X(t)X(0)^{-1}$$

- The solution of a non-homogenous differential equation
$$\mathbf{x}' + B\mathbf{x} = \mathbf{g}(t)$$
can be found by using an integrating factor e^{Bt}. That is, multiplying by e^{Bt} and noticing that $\frac{d}{dt}(e^{Bt}\mathbf{x}) = e^{Bt}\mathbf{x}' + Be^{Bt}\mathbf{x}$, the equation becomes $\frac{d}{dt}(e^{Bt}\mathbf{x}) = e^{Bt}\mathbf{g}(t)$. Integrating both sides and then multiplying by e^{-Bt} we get
$$\mathbf{x}(t) = e^{-Bt}\int e^{Bt}\mathbf{g}(t)\,dt$$

Diagonalization and Jordan form. Some of you have asked how the methods we've discussed relate to the ones in the book based on "diagonalization". Here are some comments (This is extra-curricular, so skip it if you want!) A matrix A is **diagonalizable** if there is a matrix T such that $D = T^{-1}AT$ is a diagonal matrix. It is a basic theorem from linear algebra that an $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors $v^{(1)}, \ldots, v^{(n)}$, in which case we can take T to be the matrix whose columns are $v^{(1)}, \ldots, v^{(n)}$. Then, if we define $\mathbf{y} = T^{-1}\mathbf{x}$, then $\mathbf{x}' = A\mathbf{x}$ can be rewritten as $\mathbf{y}' = T^{-1}AT\mathbf{y} = D\mathbf{y}$. This is very easy to solve, because e^{Dt} is easy to calculate for a diagonal matrix. If a matrix does *not* have n linearly independent eigenvectors, this same method almost works: by taking T to be a matrix whose columns are n linearly independent eigenvectors or *generalized* eigenvectors, it is possible to transform the matrix into what is called **Jordan canonical form**—$T^{-1}AT$ is *nearly* diagonal, except that it may have some '1's directly above the main diagonal. Luckily, it is still fairly easy to calculate e^{Jt} when J is a matrix in Jordan canonical form. Note that these methods require almost exactly the same work as the methods I showed you: you still have to calculate the eigenvalues and (generalized) eigenvectors!