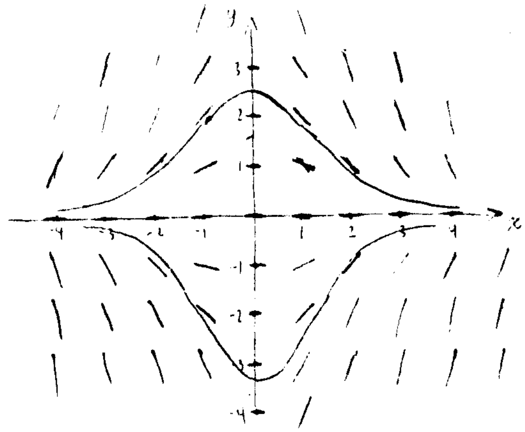
1. (5 points) Use the grid provided to draw a direction field for the equation

$$\frac{dy}{dx} = -\frac{xy}{2}$$

and use this to sketch the graphs of 2 solutions. Be sure to label your axes appropriately, and show all four quadrants.



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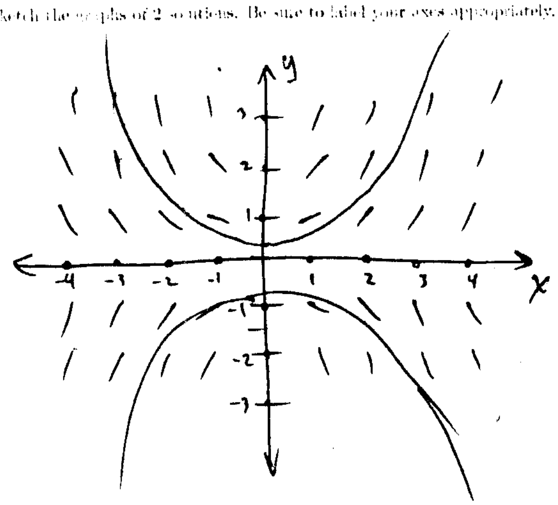
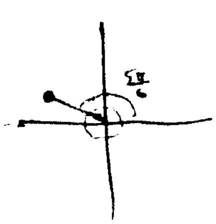


2. (5 points) Write the number $-5\sqrt{3}+5i$ in the form $\rho e^{i\theta}$, where $\rho > 0$ and θ is real.

$\rho = \sqrt{(5\sqrt{3})^2 + 5^2} = \sqrt{75+25} = 10$

$-5\sqrt{3} + 5i = 10(\cos\theta + i\sin\theta) \implies \frac{-\sqrt{3}}{2} = \cos\theta, \quad \frac{1}{2} = \sin\theta$

$\implies \theta = \frac{5\pi}{6}$



So $\boxed{-5\sqrt{3} + 5i = 10e^{\frac{5\pi}{6}i}}$

2. (5 points) Write the number $-5\sqrt{3}-5i$ in the form $\rho e^{i\theta}$, where $\rho > 0$ and θ is real.

Similar but $-5\sqrt{3} - 5i = 10e^{\frac{7\pi}{6}i}$

3. (10 points) Find the general solution for two of the following three equations:

$y' - y = t \qquad y' - y^2 = t \qquad y' = y^2 + y$

(If you try more than two of them, you must clearly indicate which two you want graded!)

$$y' - y = t$$

linear, so use an integrating factor $\mu(t) = e^{-t}$

$$y'e^{-t} - ye^{-t} = te^{-t}$$

$$\frac{d}{dt}(ye^{-t}) = te^{-t}$$

$$ye^{-t} = \int te^{-t}\,dt$$

use integration by parts.

$$= -(1+t)e^{-t} + c$$

So $\boxed{y = ce^{+t} - (1+t)}$

This is really just the logistic equation. It is separable:

$$\frac{dy}{(y+1)y} = dt$$

$$\int \frac{dy}{(y+1)y} = t + c$$

Integrate this using partial fractions:

$$\int \frac{dy}{(y+1)y} = \int \left(\frac{-1}{y+1} + \frac{1}{y} \right) dy$$

$$= \ln y - \ln(y+1)$$

$$= \ln \frac{y}{y+1}$$

So

$$\ln \frac{y}{y+1} = t + c$$

$$\frac{y}{y+1} = e^{t+c}$$

$$y = e^{t+c}(y+1)$$

$$y(1 - e^{t+c}) = e^{t+c}$$

$$y = \frac{e^{t+c}}{1 - e^{t+c}}$$

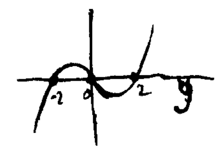
$$= \frac{ae^t}{1 - ae^t} \quad (\text{where } a = e^c)$$

4. (5 points) Consider the differential equation $y' = y^3 - 2y$. Without solving it, determine the equilibrium solutions, and sketch several solutions.

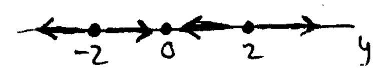
$$y^3 - 2y = y(y^2-2) \text{ is zero when } y = 0 \text{ or } \pm 2$$

Since $(1)^3 - 2(1) = -1 < 0$ and $(-1)^3 - 2(-1) = 1 > 0$,

the graph of $f(y) = y^3 - 2y$ looks like



So the phase line looks like:



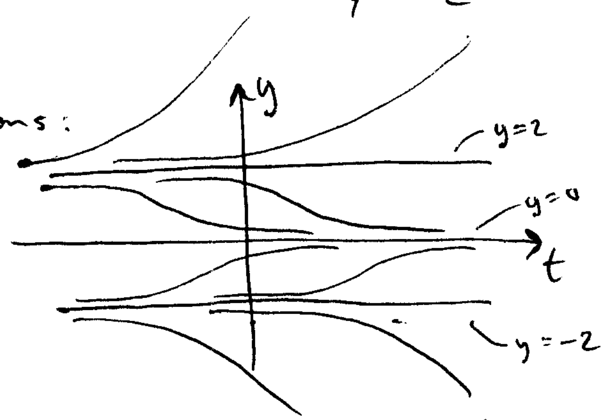
The equilibrium solutions are:

$y = -2$ unstable

$y = 0$ stable

$y = 2$ unstable

Sketch of solutions:



5. (5 points) You start with 100 grams of some radioactive substance, but one week later you only have 80 grams left. How much will be left after t weeks?

Solution of the decay equation is $Q = Q_0 e^{-rt} = 100e^{-rt}$.

When $t = 1$, we have

$$Q(1) = 100e^{-r(1)} = 80$$

$$\Rightarrow e^{-r(1)} = \frac{4}{5} \quad \Rightarrow -r = \ln\left(\frac{4}{5}\right)$$

$$r = \ln\left(\frac{5}{4}\right)$$

So $Q(t) = 100e^{\ln(\frac{4}{5})t}$

$$= \boxed{100 \cdot \left(\frac{4}{5}\right)^t}$$

6. (5 points) Solve the initial value problem:

$$4y'' + 4y' + y' = 0 \qquad y(0) = 2 \qquad y'(0) = 1.$$

Guess $y = e^{rt} \Longrightarrow 4r^2 + 4r + 1 = 0$

$$(2r+1)^2 = 0 \qquad \text{so } r = -\frac{1}{2}$$

Since there's only one root, the general solution is

$$y = c_1 e^{-\frac{1}{2}t} + c_2 t e^{-\frac{1}{2}t}$$

$$y(0) = c_1 = 2$$

$$y'(t) = -\frac{c_1}{2} e^{-\frac{1}{2}t} + \frac{-c_2}{2} t e^{-\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}$$

$$y'(0) = -\frac{c_1}{2} + c_2 = 1$$

$$\Rightarrow c_2 = 1 + \frac{c_1}{2} = 1 + \frac{2}{2} = 2$$

So $y = 2e^{-\frac{1}{2}t} + 2te^{-\frac{1}{2}t}$

$$= \boxed{2e^{-\frac{1}{2}t}(1+t)}$$

7. (5 points) You hang a 1-kilogram mass on a spring and it stretches 0.49 meters in equilibrium. You then pull the mass down half a meter from its equilibrium position and give it a downward shove with an initial speed of 1 meter per second. Assuming that there is a damping constant of 4 Newton-seconds per meter, calculate the position of the mass as a function of time. *[Recall that the acceleration of gravity in these units is $g = 9.8$ meters per second squared.]*

The differential equation for these spring problems is

$$my'' + cy' + ky = 0$$

$m = 1$ kilogram
$c = 4$ N-s/meter

To find k, note that at equilibrium

$$mg = k\,\Delta \ell$$

$$(1\,\text{kg})(9.8\,\text{m/s}^2) = k \cdot .49\,\text{m}$$

$$\text{So} \quad k = \frac{9.8}{.49} = 20 \quad (\text{kg/s}^2)$$

So the diff. eq. is $\quad 1y'' + 4y' + 20y = 0$.

Solve this using the initial conditions given in the problem, namely $y(0) = -\frac{1}{2}$ (meter), $y'(0) = -1$ (meters/sec).

I'll leave that to you ... !