

Math 21C
Final Exam
7/31/08

Name:

Signature:

Student ID:

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 100 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. Definitions and Examples: consider a smooth function $f(x, y, z)$.
- a. (2 points) Write the definition of the *partial derivative*, $\frac{\partial f}{\partial y}$. (an example is **not** sufficient for full credit)
- b. (2 points) Write the definition of the *gradient*, ∇f . (an example is **not** sufficient for full credit)
- c. (2 points) Write the definition of the *directional derivative* of f at the point \mathbf{c} in the direction of \mathbf{v} , $(D_{\mathbf{v}}f)(\mathbf{c})$. (an example is **not** sufficient for full credit)
- d. (2 points) Write the definition of the *Hessian*, $H(f)$. (an example is **not** sufficient for full credit)
- e. (2 points) Consider a sequence a_n . Write the "epsilon" definition for the following statement: "the sequence a_n converges to L ", i.e. $\lim_{n \rightarrow \infty} a_n = L$. (an example is **not** sufficient for full credit)

2. Short Answers: let $f(x, y, z) = 2x - y + z \cos z$

a. (5 points) Compute ∇f , $H(f)$, $\frac{\partial^3 f}{\partial x \partial y \partial z}$, and the directional derivative of f at the point $(0, 0, 0)$ in the direction of $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b. (5 points) Let $x(s, t, p) = s + t$, $y(s, t, p) = p^2 - t$, and $z(s, t, p) = 2$. Calculate $\frac{\partial f}{\partial t}$.

3. a. (5 points) Write an equation for the plane tangent to the surface $x^3 - y^2 + z = -1$ at the point $(0, 1, 0)$.

b. (5 points) Write an equation for the line orthogonal to the surface through the point $(-1, 1, 1)$.

4. a. (5 points) Find and classify all critical points for the function $f(x, y) = x^4 - x^2 - y^2$.

b. (5 points) Find all local extrema for this function subject to the constraint $x = y$

5. Consider the trajectory $\mathbf{p}(t) = \begin{pmatrix} 1 - 2t^{-2} \\ 1 + t^{-2} \\ 3t^{-2} \end{pmatrix}$, $1 < t < \infty$.

a. (5 points) Calculate the arclength of this path.

b. (5 points) Let $\theta(t)$ denote the angle between the trajectory's velocity, written $\mathbf{p}'(t)$, and the vector $\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$. Compute $\lim_{t \rightarrow \infty} \theta(t)$

6. (10 points) Write an equation for the plane containing the following two lines: one line is orthogonal to the surface $x + y - z^2 = 1$ through $(1, 1, 1)$, while the other line is parallel to the line of intersection of $x + y = 1$ with $x - y = 0$, and also contains the point $(1, 1, 1)$.

7. Determine whether the following sequences or series converge or diverge:

a. (2 points) $a_n = n^{-2}$.

b. (2 points) $b_n = \frac{1-n}{1+n^2}$.

c. (3 points) $\sum_{n=0}^{\infty} \frac{n}{n+1}$.

d. (3 points) $\sum_{n=0}^{\infty} \frac{n}{n^3+1}$.

8. a. (5 points) Provide examples of *nonzero* vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} such that $\mathbf{u} \times \mathbf{w} = \mathbf{v} \times \mathbf{w}$, but $\mathbf{u} \neq \mathbf{v}$.

b. (5 points) Provide examples of *nonzero* vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} such that $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$, but $\mathbf{u} \neq \mathbf{v}$.

9. a. (5 points) Prove that the diagonals of a parallelogram are orthogonal exactly when the parallelogram is a rhombus. (Hint: you need to prove two things - rhombus' diagonals are orthogonal, and non-rhombus' diagonals are not).

b. (5 points) Prove or disprove: if $a_n > 0$ for all n , and $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n^2$ also converges.

10. Consider vector-valued functions (with three components) $\mathbf{u}(t)$ and $\mathbf{v}(t)$. Prove or disprove the following:

a. (5 points) $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$

b. (5 points) $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) - \mathbf{u}(t) \times \mathbf{v}'(t).$

Bonus. (10 points) Given a vector-valued function

$$\mathbf{V}(x, y, z) = \begin{pmatrix} V_x(x, y, z) \\ V_y(x, y, z) \\ V_z(x, y, z) \end{pmatrix}, \text{ the } \textit{curl} \text{ of } \mathbf{V}, \text{ written } \nabla \times \mathbf{V},$$

is defined similarly to the cross product:

$$\nabla \times \begin{pmatrix} V_x(x, y, z) \\ V_y(x, y, z) \\ V_z(x, y, z) \end{pmatrix} = \begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix}. \text{ Prove the following:}$$

The curl of any gradient is zero, i.e. if $f(x, y, z)$ is a smooth scalar-valued function, then $\nabla \times (\nabla f) = \mathbf{0}$. (Warning: it is not necessarily true (unless you can prove) that $\nabla \times (\nabla f) = (\nabla \times \nabla)f$.)