Math 21C
Final Exam
7/31/08

Name: Solutions

Signature:

Student ID:

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 100 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. Definitions and Examples: consider a smooth function $f(x, y, z)$.

a. (2 points) Write the definition of the *partial derivative*, $\frac{\partial f}{\partial y}$. (an example is **not** sufficient for full credit)

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y}$$

b. (2 points) Write the definition of the *gradient*, ∇f. (an example is **not** sufficient for full credit)

$$\vec{\nabla} f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

c. (2 points) Write the definition of the *directional derivative* of f at the point $\mathbf{c}$ in the direction of $\mathbf{v}$, $(D_{\mathbf{v}}f)(\mathbf{c})$. (an example is **not** sufficient for full credit)

$$(D_{\hat{v}} f)(\vec{c}) = \left(\vec{\nabla} f\big|_{\vec{c}}\right) \cdot \hat{v}$$

d. (2 points) Write the definition of the *Hessian*, $H(f)$. (an example is **not** sufficient for full credit)

$$Hf = \begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y & \partial^2 f/\partial x \partial z \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 & \partial^2 f/\partial y \partial z \\ \partial^2 f/\partial z \partial x & \partial^2 f/\partial z \partial y & \partial^2 f/\partial z^2 \end{pmatrix}$$

e. (2 points) Consider a sequence a_n. Write the "epsilon" definition for the following statement: "the sequence a_n converges to L", i.e. $\lim_{n\to\infty} a_n = L$. (an example is **not** sufficient for full credit)

For all $\varepsilon > 0$, there exists an N such that

$$n > N \Rightarrow |a_n - L| < \varepsilon.$$

2. Short Answers: let $f(x, y, z) = 2x - y + z\cos z$

a. (5 points) Compute ∇f, $H(f)$, $\frac{\partial^3 f}{\partial x \partial y \partial z}$, and the directional derivative of f at the point $(0,0,0)$ in the direction of $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$\vec{\nabla} f = \begin{pmatrix} 2 \\ -1 \\ \cos z - z\sin z \end{pmatrix} \qquad Hf = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-2\sin z - z\cos z) \end{pmatrix}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x} \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial x} 0 = 0.$$

$$\left(D_{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} f\right)(\vec{0}) = \left(\vec{\nabla} f\big|_{\vec{0}}\right) \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{3}} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} / \sqrt{3} = 0$$

b. (5 points) Let $x(s,t,p) = s + t$, $y(s,t,p) = p^2 - t$, and $z(s,t,p) = 2$. Calculate $\frac{\partial f}{\partial t}$.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= 2(1) + (-1)(-1) + (\cos z - z\sin z)0$$

$$= 2 + 1 = 3.$$

3. a. (5 points) Write an equation for the plane tangent to the surface $F = x^3 - y^2 + z = -1$ at the point $(0, 1, 0)$.

$$\vec{\nabla} F = \begin{pmatrix} 3x^2 \\ -2y \\ 1 \end{pmatrix} \qquad \vec{\nabla} F \Big|_{(0,1,0)} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \vec{n}$$

$$0x - 2y + z = d$$

$$\cancel{2(-2)}$$

$$-2(1) + 0 = -2 = d$$

$$\boxed{-2y + z = -2}$$

b. (5 points) Write an equation for the line orthogonal to the surface through the point $(-1, 1, 1)$.

$$\vec{\nabla} F \Big|_{(0,1,0)} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \vec{m}$$

$$\boxed{\vec{p}(t) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} t \qquad -\infty < t < \infty}$$

4. a. (5 points) Find and classify all critical points for the function $f(x,y) = x^4 - x^2 - y^2$.

$$\vec{\nabla} f = \begin{pmatrix} 4x^3 - 2x \\ -2y \end{pmatrix} = \vec{0} \Rightarrow \begin{array}{r} 2x(2x^2-1) = 0 \\ -2y = 0 \end{array}$$

$$\Rightarrow 3 \text{ CPs @ } (0,0), (\sqrt{1/2}, 0), (-\sqrt{1/2}, 0)$$

$$\det(Hf) = \det\begin{pmatrix} 12x^2 - 2 & 0 \\ 0 & -2 \end{pmatrix} = 4 - 24x^2 = D$$

$$D(0,0) = 4 > 0, \ \frac{\partial^2 f}{\partial y^2} = -2 < 0 \Rightarrow \underline{(0,0) \text{ is a max.}}$$

$$D(\sqrt{1/2}, 0) = D(-\sqrt{1/2}, 0) = -8 < 0$$

$\Rightarrow$ $(\sqrt{1/2}, 0)$, $(-\sqrt{1/2}, 0)$ are saddles

b. (5 points) Find all local extrema for this function subject to the constraint $x = y$

$$g(x,y) = x - y = 0$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\begin{pmatrix} 4x^3 - 2x \\ -2y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} \lambda = 2y \\ \lambda = 4x^3 - 2x \\ x = y \end{cases}$$

$$\Rightarrow 4x^3 - 2x = 2x \Rightarrow x^3 = x$$

$$\Rightarrow x = 0 \text{ or } x = \pm 1$$

$$\Rightarrow 3 \text{ CPs @ } (0,0), (1,1), (-1,-1).$$

5. Consider the trajectory $\mathbf{p}(t) = \begin{pmatrix} 1 - 2t^{-2} \\ 1 + t^{-2} \\ 3t^{-2} \end{pmatrix}$, $1 < t < \infty$.

a. (5 points) Calculate the arclength of this path.

$$L = \int_1^\infty \|\vec{P}'(t)\|\,dt = \int_1^\infty \left\| \begin{pmatrix} 4t^{-3} \\ -2t^{-3} \\ -6t^{-3} \end{pmatrix} \right\| dt$$

$$= \int_1^\infty t^{-3} \left\| \begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix} \right\| dt = \int_1^\infty \sqrt{56}\, t^{-3}\, dt$$

$$= \lim_{b\to\infty} \sqrt{56}\, \frac{t^{-2}}{-2}\Big|_1^b = \lim_{b\to\infty} \left(\sqrt{56} \left(-\frac{b^{-2}}{2} + \frac{1}{2} \right) \right)$$

$$= \boxed{\frac{1}{2}\sqrt{56}}$$

b. (5 points) Let $\theta(t)$ denote the angle between the trajectory's velocity, written $\mathbf{p}'(t)$, and the vector $\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$. Compute $\lim_{t\to\infty} \theta(t)$

$$\vec{P}'(t) = \begin{pmatrix} 4t^{-3} \\ -2t^{-3} \\ -6t^{-3} \end{pmatrix}$$

$$\lim_{t\to\infty} \theta(t) = \lim_{t\to\infty} \operatorname{ArcCos}\left(\frac{\vec{P}' \cdot \vec{v}}{\|\vec{P}'\|\|\vec{v}\|} \right) = \operatorname{ArcCos} \lim_{t\to\infty} \left(\frac{-42t^{-3}}{t^{-3}\sqrt{56}\sqrt{38}} \right)$$

$$= \operatorname{ArcCos} \lim_{t\to\infty} \left(\frac{-42}{\sqrt{56}\sqrt{38}} \right)$$

$$\boxed{= \operatorname{ArcCos}\left(\frac{-42}{\sqrt{56}\sqrt{38}} \right)}$$

f

6. (10 points) Write an equation for the plane containing the following two lines: one line is orthogonal to the surface $x+y-z^2=1$ through $(1,1,1)$, while the other line is parallel to the line of intersection of $x+y=1$ with $x-y=0$, and also contains the point $(1,1,1)$.

$$\vec{\nabla} f = \begin{pmatrix} 1 \\ 1 \\ -2z \end{pmatrix} \quad \vec{\nabla} f \Big|_{(1,1,1)} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \vec{m}_1$$

- $$\vec{P}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} t$$

$$\vec{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \vec{m}_2$$

- $$\vec{P}_2(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} t$$

Plane contains $\vec{P}_1, \vec{P}_2$ so is parallel to $\vec{m}_1, \vec{m}_2$. then

$$\vec{n} = \vec{m}_1 \times \vec{m}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$-2x + 2y + 0z = d$$
$$-2(1) + 2(1) = 0 = d$$

$$\boxed{-2x + 2y = 0}$$

7. Determine whether the following sequences or series converge or diverge:

a. (2 points) $a_n = n^{-2}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0 \Rightarrow a_n \to 0.$$

b. (2 points) $b_n = \frac{1-n}{1+n^2}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1-n}{1+n^2} = 0 \Rightarrow b_n \to 0.$$

c. (3 points) $\sum_{n=0}^{\infty} \frac{n}{n+1}$.

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{n}{n+1}$$ diverges (Nth term test)

d. (3 points) $\sum_{n=0}^{\infty} \frac{n}{n^3+1}$. LCT vs. $\frac{1}{n^2}$

recall, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. $(P=2>1)$

$$\lim_{n\to\infty} \frac{\frac{n}{n^3+1}}{1/n^2} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1,$$ so both series do the same. hence,

$$\sum_{n=0}^{\infty} \frac{n}{n^3+1}$$ converges.

8. a. (5 points) Provide examples of *nonzero* vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\mathbf{u} \times \mathbf{w} = \mathbf{v} \times \mathbf{w}$, but $\mathbf{u} \neq \mathbf{v}$.

let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

clearly, $\vec{u} \neq \vec{v}$. however,

$\vec{v} = 2\vec{u} = 2\vec{w}$, i.e. $\vec{u} \parallel \vec{v} \parallel \vec{w}$, so

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w} = \vec{v} \times \vec{w} = \vec{0}$.

b. (5 points) Provide examples of *nonzero* vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$, but $\mathbf{u} \neq \mathbf{v}$.

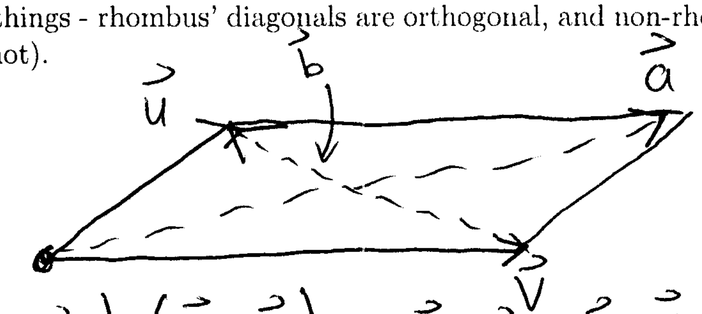
let $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$,

$\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

clearly, $\vec{u} \neq \vec{v}$. however,

$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$.

9. a. (5 points) Prove that the diagonals of a parallelogram are orthogonal exactly when the parallelogram is a rhombus. (Hint: you need to prove two things - rhombus' diagonals are orthogonal, and non-rhombus' diagonals are not).



$\vec{a} = \vec{u} + \vec{v}$

$\vec{b} = \vec{u} - \vec{v}$

$\vec{a} \cdot \vec{b} = (\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v}$

$= \|\vec{u}\|^2 - \|\vec{v}\|^2$. hence $\vec{a} \cdot \vec{b} = 0$ (diags orthogonal)

iff $\|\vec{u}\| = \|\vec{v}\|$, i.e.  is a rhombus.

b. (5 points) Prove or disprove: if $a_n > 0$ for all n, and $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n^2$ also converges.

LCT: since $\sum_{n=0}^{\infty} a_n$ converges, $a_n \to 0$.

then $\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n = 0$

$\Rightarrow$ since $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} a_n^2$ also converges (limit comparison)

10. Consider vector-valued functions (with three components) $\mathbf{u}(t)$ and $\mathbf{v}(t)$. Prove or disprove the following:

a. (5 points) $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.

Proof: $\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \frac{d}{dt}(u_1 v_1 + u_2 v_2 + u_3 v_3)$

$= (u_1' v_1 + u_1 v_1' + u_2' v_2 + u_2 v_2' + u_3' v_3 + u_3 v_3')$

$= (\vec{u}' \cdot \vec{v}) + (\vec{u} \cdot \vec{v}')$,

b. (5 points) $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) - \mathbf{u}(t) \times \mathbf{v}'(t)$.

Disproof: (should be +)

Counterexample

let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix}$.

then $\frac{d}{dt}(\vec{u} \times \vec{v}) = \frac{d}{dt}\begin{pmatrix} +t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} +1 \\ 1 \\ 0 \end{pmatrix}$, but

$\vec{u}' \times \vec{v} - \vec{u} \times \vec{v}' = \vec{0} \times \vec{v} - \vec{u} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -\vec{u} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

argh!!

bad CX, good strategy

instead, one should use $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$

Bonus. (10 points) Given a vector-valued function

$$\mathbf{V}(x,y,z) = \begin{pmatrix} V_x(x,y,z) \\ V_y(x,y,z) \\ V_z(x,y,z) \end{pmatrix}, \text{ the } \textit{curl} \text{ of } \mathbf{V}, \text{ written } \nabla \times \mathbf{V},$$

is defined similarly to the cross product:

$$\nabla \times \begin{pmatrix} V_x(x,y,z) \\ V_y(x,y,z) \\ V_z(x,y,z) \end{pmatrix} = \begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix}. \text{ Prove the following:}$$

The curl of any gradient is zero, i.e. if $f(x,y,z)$ is a smooth scalar-valued function, then $\nabla \times (\nabla f) = \mathbf{0}$. (Warning: it is not necessarily true (unless you can prove) that $\nabla \times (\nabla f) = (\nabla \times \nabla) f$.)

$$\vec{\nabla} \times (\vec{\nabla} f) = \vec{\nabla} \times \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y} \\ \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z} \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}.$$

(mixed partials are equal for smooth functions)

$$\text{"} \vec{\nabla} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \text{"}$$