Math 21C
Midterm 1
7/3/08

Name: Solutions

Signature:

Student ID:

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 50 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. Definitions and Examples:

a. (2 points) Write the definition of a *Geometric Series.* (an example is **not** sufficient for full credit)

$$\sum_{n=0}^{\infty} r^n$$

b. (2 points) Write the definition of an *Alternating Series.* (an example is **not** sufficient for full credit)

$$\sum_{n=1}^{\infty} (-1)^n u_n \; ; \; u_n \geq 0$$

c. (2 points) Write the definition of a *p-series.* (an example is **not** sufficient for full credit)

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

d. (2 points) Write the definition of a *power series centered at c.* (an example is **not** sufficient for full credit)

$$\sum_{n=0}^{\infty} u_n (x-c)^n$$

e. (2 points) Let $f = f(x)$. Write the definition of the *Taylor series* for f centered at c. (an example is **not** sufficient for full credit)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

2. Short Answers

a. (5 points) State the *ratio test* for infinite series.

if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

b. (5 points) State the *nth-term test* for infinite series.

if $\lim_{n\to\infty} a_n \neq 0$,

then $\sum_{n=1}^{\infty} a_n$ diverges

3. Determine whether the following sequences converge or diverge. If a sequence converges, find its limit.

a. (3 points) $a_n = \frac{n+1}{n-1}$.

$$\lim_{n\to\infty} \frac{n+1}{n-1} = 1 \quad \text{(leading coefficients)}$$

$$\Rightarrow \boxed{a_n \text{ converges to } 1}$$

b. (3 points) $b_n = 1 + e^{\frac{\sin n}{n}}$.

$$\lim_{n\to\infty}\left(1 + e^{\frac{\sin n}{n}}\right) = 1 + \lim_{n\to\infty} e^{\frac{\sin n}{n}} \quad \text{(addition)}$$

$$= 1 + e^{\lim_{n\to\infty}\frac{\sin n}{n}} = 1 + e^0 = 2$$

(continuity)

$$\Rightarrow \boxed{b_n \text{ converges to } 2}$$

c. (4 points) $a_n = \frac{(2n-1)!}{2(n-1)!}$.

$$\lim_{n\to\infty} \frac{(2n-1)!}{2(n-1)!} = \frac{1}{2}\lim_{n\to\infty} \frac{(2n-1)(2n-2)\ldots(n)(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{(n-1)(n-2)\ldots 3\cdot 2\cdot 1}$$

$$= \frac{1}{2}\lim_{n\to\infty}(2n-1)(2n-2)\ldots(n+1)n = \infty$$

$$\Rightarrow \boxed{c_n \text{ diverges}}$$

4. Determine whether the following series converge or diverge. If a geometric series converges, find its limit. If convergence depends on a constant input, provide conditions on this constant.

a. (5 points) $\sum_{n=1}^{\infty} \frac{1}{p^n}$. $= \sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n$ is geometric,

with ratio $1/p$ so $\boxed{\text{converges if } \left|\frac{1}{p}\right| < 1}$ and diverges otherwise

$$\sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n = \sum_{n=0}^{\infty} \frac{1}{p}\left(\frac{1}{p}\right)^n = \frac{1}{p}\sum_{n=0}^{\infty}\left(\frac{1}{p}\right)^n = \frac{1/p}{1-1/p}$$

$$= \frac{1}{p-1} \Rightarrow \boxed{\text{converges to } \frac{1}{p-1} \text{ (when it converges)}}$$

b. (5 points) $\sum_{n=1}^{\infty} c - \frac{n}{n+1}$.

$$= \sum_{n=1}^{\infty} \frac{c(n+1)-n}{n+1} = \sum_{n=1}^{\infty} \frac{(c-1)n+c}{n+1}$$

if $\boxed{c \neq 1, \text{ then}}$ $\lim_{n\to\infty} \frac{(c-1)n+c}{n+1} = c-1 \neq 0$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} c - \frac{n}{n+1} \text{ diverges}}$$

if $c=1$, $\sum_{n=1}^{\infty} \frac{c}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$

$= -1 + \sum_{n=1}^{\infty} \frac{1}{n}$, which is harmonic,

so diverges. hence, if $\boxed{c=1, \sum_{n=1}^{\infty} c - \frac{n}{n+1} \text{ diverges}}$

5. (10 points) Determine the values of x for which the following series converges conditionally, converges absolutely, or diverges. What are the center, radius, and interval of convergence?

$$S = \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{(n+2)2^{n-1}}.$$ is a power series: ratio test!

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x+1)^{n+2} / (n+3)2^n}{(x+1)^{n+1} / (n+2)2^{n-1}} \right|$$

$$= \lim_{n\to\infty} |x+1| \frac{n+2}{2(n+3)} = \frac{|x+1|}{2} < 1$$ for convergence (absolute)

$-3 < x < 1$

> 1 for divergence

endpoints:

$x = -3$

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{(n+2)2^{n-1}} = \sum_{n=0}^{\infty} 4(-1)^{n+1} \frac{1}{n+2}$$

$$= 4 \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n},$$ a multiple of the alternating harmonic series, conv. conditionally

$x = 1$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{(n+2)2^{n-1}} = 4 \sum_{n=2}^{\infty} \frac{1}{n} = 4\left(-1 + \sum_{n=1}^{\infty} \frac{1}{n}\right),$$ the harmonic series, which diverges.

So: S converges absolutely if $-3 < x < 1$,
S converges conditionally if $x = -3$,
and S diverges otherwise

6. (10 points) Compute (no shortcuts!!) the *Taylor series* centered at 0 for the function

$$f(x) = e^{1-\frac{3x}{4}}.$$

k	$f^{(k)}(x)$	$f^{(k)}(c)$
0	$e^{1-3x/4}$	|
1	$-\frac{3}{4}e^{1-3x/4}$	|
2	$\left(-\frac{3}{4}\right)^2 e^{1-3x/4}$	|
3	$\left(-\frac{3}{4}\right)^3 e^{1-3x/4}$	|
$\vdots$	$\vdots$	
n	$\left(-\frac{3}{4}\right)^n e^{1-3x/4}$	$(-3/4)^n e^1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-3/4)^n e}{n!}(x-c)^n$$

7. (10 points) Using what you know about familiar Taylor series, write a power series for arctan $(2x)$. For what values of x does your series converge?

(Hint: you should be familiar with the series for $\frac{1}{1-x}$.)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (\text{if } |x| < 1)$$

So $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

then $$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt$$

$$= \int_0^x \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \int_0^x x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

hence, $$\boxed{\arctan(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} \quad \text{if } |x| < 1}$$

8. (10 points) Using what you know about familiar Taylor series, write a power series for $f(x) = \sin(2x)$. Similarly, write the the first four terms of a series for $g(x) = (\sin x)(\cos x)$. For what values of x will these series converge? Use this to compute

$$\lim_{x\to 0}[f(x) - g(x)]$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\text{so } \sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$$

$$\text{also, } \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\text{so } \sin x \cos x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= x + x^3\left(-\frac{1}{3!} - \frac{1}{2!}\right) + x^5\left(\frac{1}{5!} + \frac{1}{3!\,2!} + \frac{1}{4!}\right)$$

$$+ x^7\left(-\frac{1}{7!} - \frac{1}{5!\,2!} - \frac{1}{3!\,4!} - \frac{1}{6!}\right) + \cdots$$

$$\text{then } f(x) - g(x) = x(2-1) + x^3\left(-\frac{2^3}{3!} + \frac{1}{3!} + \frac{1}{2!}\right) + \cdots$$

$$\text{so } \lim_{x\to 0}\left(f(x) - g(x)\right) = 0.$$

9. (10 points) Willy Wonka's newly patented *irresistable gobstopper* is highly addictive. There are currently 1000 gobstopper addicts. Every year, the number of addicts triples due to advertising, but only one tenth of these people survive schnozberry-extract withdrawal symptoms. Those who survive are drastically changed (so count as different people the next year). Write an infinite series which represents the total number of gobstopper addicts throughout the future. Will gobstopper addiction lead to extinction of the human race?

let a_n denote

\# addicts in year n

(after now)

then $a_0 = 1000$,

$$a_n = \left(\frac{3}{10}\right)^n \cdot 1000$$

and total: $\boxed{\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1000 \left(\frac{3}{10}\right)^n}$

$= 1000 \sum_{n=0}^{\infty} \left(\frac{3}{10}\right)^n$, geometric series w/ $r = 3/10 < 1$,

$$= 1000 \frac{1}{1 - 3/10} = \frac{10{,}000}{7}$$

there will only be $\frac{10{,}000}{7}$ survivors

... so what? if $\frac{9}{10}$ of addicts die, $\frac{1}{10}$ survive... that'll never cause extinction

10. (10 points) Let p_n denote the nth prime: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, etc. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{p_n^2}$$

note, $n < P_n$, so $n^2 < P_n^2$,

so $\frac{1}{n^2} > \frac{1}{P_n^2}$

then by direct comparison, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $(P=2>1)$, $\sum_{n=1}^{\infty} \frac{1}{P_n^2}$ also converges.

---

alternatively, $\sum_{n=1}^{\infty} \frac{1}{P_n^2}$ is a "sparser version" of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, where some terms become zero.

hence, again by direct comparison, $\sum_{n=1}^{\infty} \frac{1}{P_n^2}$ converges

Conclusive: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is conv. p-series $(p=2>1)$,

$\lim_{n\to\infty} \frac{1/n^3}{1/n^2} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

Inconclusive: $a_n = (-1)^n$, test does not apply!

Bonus. (10 points) State the *limit comparison test* for infinite series. Explain in words why the conclusions of this test make sense. Provide examples (1 each) of series for which this test is conclusive or inconclusive (as always, make sure to justify your claims).

if $a_n, b_n > 0$:

if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ conv.}$$

if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then

$$\sum_{n=1}^{\infty} b_n \text{ conv.} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv.}$$

if $\lim_{n\to\infty} \frac{a_n}{b_n} = \underset{(\text{non zero})}{c}$, then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ conv.}$$

---

if limit $= \infty$, then $a_n \gg b_n$, so $\sum a_n > \sum b_n$

if limit $= 0$, then $b_n \gg a_n$, and so $\sum a_n < \infty \Rightarrow \sum b_n < \infty$.

similarly, $\sum b_n < \infty \Rightarrow \sum a_n < \infty$.

if limit $= c$, then $a_n \approx c \cdot b_n \Rightarrow$ constant multiple rule says both series do same.