Math 21C
Midterm 2
7/17/08

Name: Solutions

Signature:

Student ID:

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 50 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. Definitions and Examples: let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

a. (2 points) Write the definition of $\mathbf{u} \cdot \mathbf{v}$. (an example is **not** sufficient for full credit)

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

b. (2 points) Write the definition of $\mathbf{u} \times \mathbf{v}$. (an example is **not** sufficient for full credit)

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

c. (2 points) Write the definition of $\| \mathbf{u} \|$. (an example is **not** sufficient for full credit)

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

d. (2 points) Write the definition of the projection $proj_{\mathbf{u}}\mathbf{v}$. (an example is **not** sufficient for full credit)

$$proj_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$$

e. (2 points) Write the definition of the angle θ between $\mathbf{u}$ and $\mathbf{v}$. (an example is **not** sufficient for full credit)

$$\theta = \operatorname{ArcCos}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right)$$

2. Short Answers: let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

a. (5 points) Compute $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $\mathbf{u}\cdot\mathbf{v}$, and $2\mathbf{u}$.

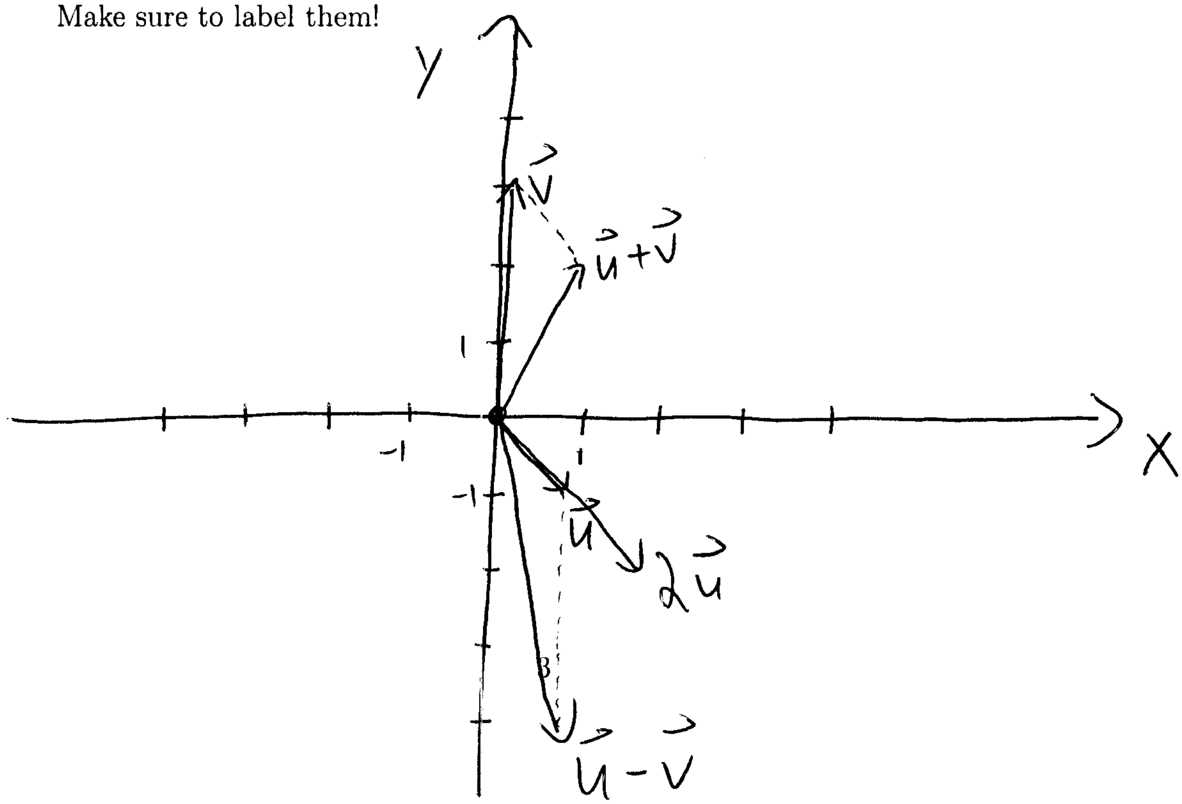
$$\vec{u}+\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+0 \\ -1+3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{u}-\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-0 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

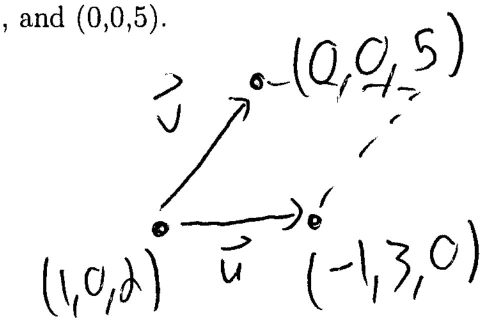
$$\vec{u}\cdot\vec{v} = 1\cdot 0 + (-1)\cdot 3 = -3$$

$$2\vec{u} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

b. (5 points) Sketch $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $2\mathbf{u}$ in the coordinate plane. Make sure to label them!



3. (10 points) Compute the area of the triangle with vertices (1,0,2), (-1,3,0), and (0,0,5).



$$\vec{u} = \begin{pmatrix}-1\\3\\0\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\3\\-2\end{pmatrix}$$

$$\vec{v} = \begin{pmatrix}0\\0\\5\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\3\end{pmatrix}$$

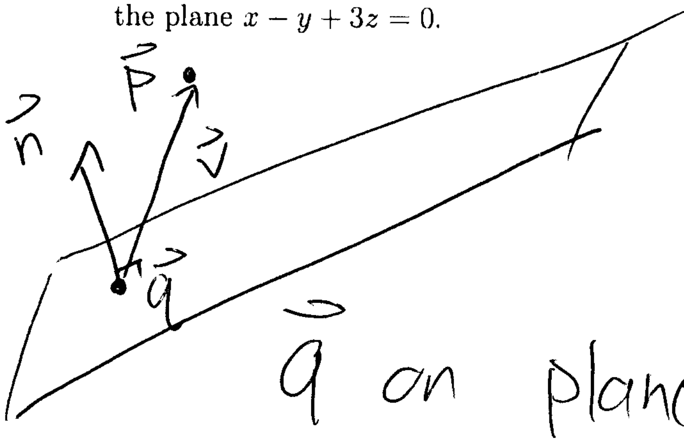
$$A = \frac{\|\vec{u} \times \vec{v}\|}{2}$$

$$= \frac{\left\| \begin{pmatrix}-2\\3\\-2\end{pmatrix} \times \begin{pmatrix}-1\\0\\3\end{pmatrix} \right\|}{2} = \frac{\left\| \begin{pmatrix}9\\8\\3\end{pmatrix} \right\|}{2}$$

$$= \frac{1}{2}\sqrt{81+64+9} = \frac{1}{2}\sqrt{154}$$

$$= \boxed{\sqrt{77/2}}$$

4. (10 points) Compute the distance between the point (0,1,-1) and the plane $x - y + 3z = 0$.



$\vec{p}$ off plane:

$$\vec{p} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\vec{q}$ on plane: $0 - 0 + 3(0) = 0$

$$\vec{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v} = \vec{p} - \vec{q} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

(normal)

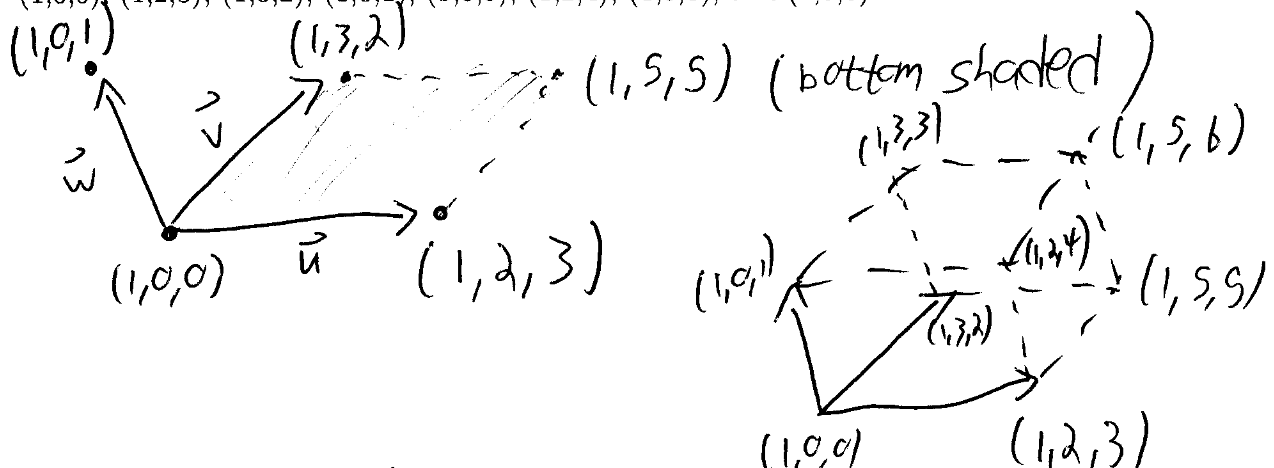
$$d = \| \mathrm{proj}_{\vec{n}} \vec{v} \|$$

$$= \left\| \frac{\vec{v} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right\| = \left\| \frac{-4}{11} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\|$$

$$\boxed{= \frac{4}{11} \sqrt{11}}$$

5. (10 points) Compute the volume of the parallelopiped with vertices (1,0,0), (1,2,3), (1,3,2), (1,0,1), (1,5,5), (1,2,4), (1,3,3), and (1,5,6).

$$\vec{u} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$\text{Vol} = |\vec{u} \cdot \vec{v} \times \vec{w}|$$

$$= \left| \vec{u} \cdot \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = \left| \vec{u} \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right| = |0| = \underline{0}$$

it must be flat...
ah-ha! all points are in the plane $\underline{x = 1}$

6. (10 points) Write a parameterization for the line of intersection of the planes $2x - y = 0$ and $y - z = 1$.

the line is $\perp$ ~~to~~ to both normal vectors $\vec{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$,

$\vec{n}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

so if $\vec{P}(t) = \vec{a} + \vec{b}t$, $\vec{b} = \vec{n}_1 \times \vec{n}_2$ works!

$$\vec{n}_1 \times \vec{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

look: $2(1) - 2 = 0$ and $2 - 1 = 1$,

so $(1, 2, 1)$ is in both planes!

so $\boxed{\vec{P}(t) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} t \quad -\infty < t < \infty}$

7. (10 points) Write a parameterization for the line which is parallel to the plane $x + y - 2z = 1$, perpendicular to the line $\mathbf{p}(t) = \begin{pmatrix} 1-t \\ 1+t \\ 1 \end{pmatrix}$, and contains the point (0,0,1).

$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} t$

parallel to plane is $\perp$ to normal,

$$\vec{n} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\perp \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, so take

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}.$$ then

since the line goes through (0,0,1),

$$\boxed{\vec{p}(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} t \quad -\infty < t < \infty}$$

8. (10 points) Let $\mathbf{u} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

Prove or disprove:

If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then

$\mathbf{v} = \mathbf{w}$.

$$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$$

$$\Rightarrow \vec{u} \times (\vec{v} - \vec{w}) = \vec{0}$$

$$\Rightarrow \vec{u} \parallel (\vec{v} - \vec{w})$$

$$\Rightarrow \vec{u} = t(\vec{v} - \vec{w}), \text{ some number } t.$$

then $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ $\quad (\vec{u} \neq \vec{0} \Rightarrow t \neq 0)$

$$\Rightarrow \vec{u} \cdot (\vec{v} - \vec{w}) = 0$$

$$\Rightarrow t(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = 0$$

$$\Rightarrow t \|\vec{v} - \vec{w}\|^2 = 0$$

$$\Rightarrow \|\vec{v} - \vec{w}\|^2 = 0$$

$$\Rightarrow \vec{v} - \vec{w} = \vec{0} \Rightarrow \vec{v} = \vec{w}$$

9. (10 points) Prove the parallelogram law: if **u** and **v** are vectors, then

$\| \mathbf{u}+\mathbf{v} \|^2 + \| \mathbf{u}-\mathbf{v} \|^2 = 2(\| \mathbf{u} \|^2 + \| \mathbf{v} \|^2)$

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2$$
$$= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v}$$
$$= 2\vec{u}\cdot\vec{u} + 2\vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u}$$
$$= 2(\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v})$$
$$= 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$$

10. (10 points) let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

Prove or Disprove: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

$$= -1 \begin{pmatrix} u_3 v_2 - u_2 v_3 \\ u_1 v_3 - u_3 v_1 \\ u_2 v_1 - u_1 v_2 \end{pmatrix}$$

$$= -1 \begin{pmatrix} v_2 u_3 - v_3 u_2 \\ v_3 u_1 - v_1 u_3 \\ v_1 u_2 - v_2 u_1 \end{pmatrix}$$

$$= -1 \left(\vec{v} \times \vec{u} \right).$$

Bonus. (10 points) An example of a two-dimensional plane in four-dimensional space is the set of solutions to the equations

$2y + 2w = -1$

$x + y + z + w = 1$ and $x - y + z - w = 2.$

$2x + 2z = 3$

Determine all four-dimensional vectors which are orthogonal to this plane. (Hint: start with two vectors parallel to the plane)

3 points in the plane:
(non-colinear) (x, y, z, w)

$\vec{a}$ $(1, -\frac{1}{2}, \frac{1}{2}, 0)$

$\vec{b}$ $(1, 0, \frac{1}{2}, -\frac{1}{2})$

$\vec{c}$ $(0, -2, 3/2, 3/2)$

$\vec{u} = \vec{b} - \vec{a} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ -1/2 \end{pmatrix}$

$\vec{v} = \vec{c} - \vec{a} = \begin{pmatrix} -1 \\ -3/2 \\ 1 \\ 3/2 \end{pmatrix}$

we want vectors $\perp$ both $\vec{u}, \vec{v}$:

$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \vec{u} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \vec{v} = 0 \Rightarrow$

$\frac{1}{2}b - \frac{1}{2}d = -a - \frac{3}{2}b + c + \frac{3}{2}d = 0$

$\Rightarrow b = d,\ a = c \Rightarrow$ $\boxed{\text{vectors } \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}, \text{ for any } a, b}$

(there's a <u>much</u> slicker proof – ask in OH!)