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Proving the Convolution Theorem

<u>Theorem 6.6.1</u> (*)

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)\,G(s) = \mathcal{L}\{h(t)\}, \quad s > a. \qquad (1)$$

where

$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \qquad (2)$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are known as the convolution integrals.

<u>Proof</u> $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$

Given that the definition of the Laplace Transform is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt,$$

we can use the dummy variables σ and τ to express their product as

$$F(s)G(s) = \left[\int_0^\infty f(\sigma)\,e^{-s\sigma}\,d\sigma\right]\left[\int_0^\infty g(\tau)\,e^{-s\tau}\,d\tau\right]$$

$$= \int_0^\infty \left[\int_0^\infty f(\sigma)\,e^{-s(\sigma+\tau)}\,d\sigma\right] g(\tau)\,d\tau.$$

Now use the change of variables $t = \sigma + \tau$ and $dt = d\sigma$,

$$= \int_0^\infty \left[\int_0^\infty (t-\tau)\,e^{-st}\,dt\right] g(\tau)\,d\tau$$

$$= \int_0^\infty \left[\int_0^\infty f(t-\tau)\,g(\tau)\,e^{-st}\,dt\right] d\tau.$$

We now reverse the order of integration to the region (t, τ) and get

$$= \int_0^\infty \left[\int_0^t f(t-\tau)\,g(\tau)\,e^{-st}\,d\tau\right] dt$$

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Rewriting the equation gives

$$F(s)G(s) = \int_0^\infty \left[\int_0^t f(t-\tau)\, g(\tau)\, d\tau \right] e^{-st}\, dt$$

$$= \mathcal{L}\left(\int_0^t f(t-\tau)\, g(\tau)\, d\tau \right).$$

Therefore,

$$H(s) = F(s)\, G(s) \quad \blacksquare$$

Works Cited

* Boyce, William E. DiPrima, Richard C. Elementary Differential Equations and Boundary Value Problems Eighth Edition. 2005. United States.