Using the Wronskian Test for Linear Independence To Show That A Linear Homogeneous Second Order Ordinary Differential Equation Can Only Exist In No More Than Two Dimensions

Take three functions, f, g, and h, and apply to them the Wronskian test for linear independence of three functions:

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \qquad (1)$$

If the three functions are linearly independent, $W(f,g,h) \neq 0$, then it can be proven that the three functions are a basis for the vector space of the ODE, making the dimension of the vector space three. However, if the Wronskian test is equal to zero then it is *possible1* that the three functions are linearly dependent, meaning that they are not a basis for the vector space, and that the dimension must be less than three. For the purposes of this essay we will only consider equations that are nice and smooth enough that when the Wronskian is equal to zero the functions are linearly dependent.

When we compute this 3×3 Wronskian, we obtain:

$$W(f,g,h) = f(g'h'' - h'g'') + h(f'g'' - g'f'') + g(h'f'' - f'h'')$$

or (2)

$$W(f,g,h) = f''(gh' - hg') + g''(hf' - fh') + h''(fg' - gf')$$

As our goal is to prove that linear second order homogeneous differential equations cannot have a dimension of more than two, the next natural step is to plug in linear second order homogeneous equations for the second derivative in equation (2). To do this we use the standard form of the linear ordinary second order differential equation,

$$y'' + p(t)y' + q(t)y = z(t), \qquad (3)$$

as the form of functions f, g, and h, then we solve for the second derivatives,

$$f'' = z_1(t) - q_1(t)f - p_1(t)f' \qquad (4)$$

$$g'' = z_2(t) - q_2(t)g - p_2(t)g' \qquad (5)$$

$$h'' = z_3(t) - q_3(t)h - p_3(t)h', \qquad (6)$$

and plug them in for the second derivatives in equation (2) with the following conditions:

$$p_1(t) = p_2(t) = p_3(t) = p(t) \qquad (7)$$

$$q_1(t) = q_2(t) = q_3(t) = q(t) \qquad (8)$$

$$z_1(t) = z_2(t) = z_3(t) = z(t). \qquad (9)$$

Performing this substitution, all $q(t)$ terms and all $p(t)$ terms cancel, equaling zero, and we obtain the following equation:

$$W(f,g,h) = (z(t) + q(t)f - p(t)f')(gh' - hg')$$
$$+(z(t) + q(t)g - p(t)g')(hf' - fh')$$
$$+(z(t) + q(t)h - p(t)h')(fg' - gf')$$

$$\begin{aligned} W(f,g,h) = z(t)(gh' - hg') - q(t)(fgh' - fg'h) - p(t)(f'gh' - f'g'h) \\ +z(t)(hf' - fh') - q(t)(f'gh - fgh') - p(t)(f'gh - fgh') \\ +z(t)(fg' - gf') - q(t)(fg'h - f'gh) - p(t)(fg'h' - f'gh') \end{aligned}$$

$$W(f,g,h) = z(t)(gh' - hg' + hf' - fh' + fg' - gf') \quad (10)$$

We set $z(t)$ equal to zero in order to make the equations homogeneous and we see that the Wronskian is equal to zero.

$$W(f,g,h) = 0 \; (11)$$

So, if

$$c_1 f + c_2 g + c_3 h = \overline{0}, \qquad (12)$$

where c_1, c_2, c_3 are not all zero, then f, g, and h are linearly dependent (one of them is a linear combination of the others). This means that the vector space does not have three basis vectors and its dimension is not three.

This does not (sadly) completely rule out high dimensions as well though. So, could a second order linear homogeneous ordinary differential equation be of a dimension higher than three? Taking into consideration the Wronskian using an n number of equations,

$$W(f_1, \dots, f_n) = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}, \qquad (13)$$

We can see that if we were to plug in the equations 4-6 then all terms grouped with $q(t)$ and all terms grouped with $p(t)$ will cancel just as they did with the three by three Wronskian and since $z(t)$is equal to zero then the Wronskian is zero and, for our purposes, the functions are linearly dependent and not a basis for the second order linear homogeneous differential equations.

What then is the dimension of a second order linear differential equation? Well, the only possible dimensions left are one and two. One doesn't make any sense as the second order differential equation considers more than one variable. All that is left is two. Taking the two by two Wronskian,

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf', \qquad (14)$$

Then, plugging in for the first derivative in (14) equations (4) and (5), solved for the first derivate using the conditions in equations (7), (8), and (9), we obtain:

$$W(f,g,h) = f\left(\frac{z(t) - q(t)g - g''}{p(t)}\right) - g\left(\frac{z(t) - q(t)f - f''}{p(t)}\right)$$
$$= \frac{z(t)(f - g) - fg'' + f''g}{p(t)}$$
$$= \frac{f''g - fg''}{p(t)} \neq 0; if\ p(t) \neq 0\ and\ f''g \neq fg''$$

We see that the two by two Wronskian for second order linear homogeneous ordinary differential equations has the possibility of being non-zero and therefore, for our purposes, there are only two basis vectors and the dimension of the second order linear homogeneous differential equation is two.

This has led us to a few conclusions concerning the Wronskian of linear homogeneous differential equations. The first and foremost that this essay has dealt with it the dimension of the vector space the equation is in. We have seen demonstrated that the dimension of the differential equation cannot be higher than the order of the equation. The order of the equation also determines the size of Wronskian needed to test for linear independence. If we have a differential equation to the nth degree then we use an $n \times n$ Wronskian determinate to test for linear independence with the first row being the original equation and the last row being the equation to the $n-1$ degree, excluding the nth degree.