1) (10 points) Use the Sandwich Theorem (and possibly other limit laws) to show that

$$\lim_{x\to\infty} \frac{1-\sin(x)}{x^2} = 0.$$

Note that $-1 \leq \sin x \leq 1$ for all x.

Hence $\underbrace{1-(-1)}_{2} \geq 1-\sin x \geq \underbrace{1-1}_{0}$ for all x.

Therefore $\frac{2}{x^2} \geq \frac{1-\sin x}{x^2} \geq 0$ for all $x \neq 0$

Note that

$$\lim_{x\to\infty} \frac{2}{x^2} = 2\cdot\left(\lim_{x\to\infty} \frac{1}{x}\right)^2 = 2\cdot 0^2 = 0$$

constant multiple, exponent limit laws

$\lim_{x\to\infty} \frac{1}{x} = 0$, proved in class

$$\lim_{x\to\infty} 0 = 0$$

So, by the Sandwich Theorem,

$$\lim_{x\to\infty} \frac{1-\sin x}{x^2} = 0$$ as well.

2) For each of the following functions, determine if there is a value of b for which the function is continuous. If there is such a value, compute it explicitly. If there is no such value, explain why there is no such value.

a)

$$f(x) = \begin{cases} \frac{x^3-x^2+x-1}{x-1}, & \text{if } x \neq 1 \\ b, & \text{if } x = 1 \end{cases}$$

In order for $f(x)$ to be continuous at $x=1$,

need $\lim_{x\to 1} f(x) = f(1) = b$

$$\lim_{x\to 1} \frac{x^3-x^2+x-1}{x-1} = \lim_{x\to 1} \frac{(x^2+1)\cancel{(x-1)}}{\cancel{x-1}} = 2$$

(x^2+1 is a polynomial)

So, set $\boxed{b=2}$

Note $f(x)$ is continuous everywhere else since it is of the form $\frac{\text{polynomial}}{\text{polynomial}}$ with non zero denominator if $x \neq 1$.

b) (5 points)

$$f(x) = \begin{cases} \frac{\cos x}{x}, & \text{if } x \neq 0 \\ b, & \text{if } x = 0 \end{cases}$$

In order for $f(x)$ to be continuous at $x=0$, need $\lim_{x\to 0} f(x) = f(0) = b$

$$\lim_{x\to 0^\pm} \frac{\cos x}{x} = \lim_{x\to 0^\pm} \left(-\left(\frac{1-\cos x}{x}\right) + \frac{1}{x} \right)$$

$$= 0 + \left(\lim_{x\to 0^\pm} \frac{1}{x}\right)$$

∞ from right

$-\infty$ from left

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

from formula sheet,

constant multiple limit law

sum law

$\boxed{\text{So there is no such value of } b.}$

3) a) (5 points) Write down the formal ε, δ definition of what it means for

$$\lim_{x\to 1}(3x^2+2) = 5$$

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-1| < \delta$ then

$$|3x^2+2-5| < \varepsilon.$$

b) (5 points) Find an interval (a,b) containing 1 such that if x is in (a,b) then $3x^2+2$ is in the interval $(4.5, 5.5)$. No need to simplify whatever expression you get, since you have no calculator.

$$4.5 < 3x^2 + 2 < 5.5$$

if and only if

$$2.5 < 3x^2 < 3.5$$

if and only if

$$\underbrace{\frac{2.5}{3}}_{\text{positive}} < x^2 < \underbrace{\frac{3.5}{3}}_{\text{positive}}$$

if and only if

$$\boxed{\sqrt{\frac{2.5}{3}} < x < \sqrt{\frac{3.5}{3}}} \leftarrow \text{contains 1.}$$

or

$$-\sqrt{\frac{3.5}{3}} < x < -\sqrt{\frac{2.5}{3}}$$

4) a) (3 points) Find the average rate of change of the function $f(x) = \sqrt{x}$ between $x = 1$ and $x = 3$. No need to simplify your answer.

$$\frac{\sqrt{3} - \sqrt{1}}{3 - 1} = \boxed{\frac{\sqrt{3}}{2}}$$

b) (2 points) Write down the limit you would have to compute to determine the rate of change of $f(x) = \sqrt{x}$ at $x = 2$.

$$\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$$

c) (5 points) Compute, with justification, the limit in part (b). *Hint: try multiplying top and bottom by the conjugate.*

$$\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \cdot \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}}$$

$$= \lim_{h \to 0} \frac{\cancel{2} + \cancel{h} - \cancel{2}}{\cancel{h}(\sqrt{2+h} + \sqrt{2})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$$

Limit quotient law

continuous in $(-0.5, 0.5)$ by root, sum continuity laws, so $\lim_{h \to 0}(\sqrt{2+h} + \sqrt{2}) = 2\sqrt{2}$

5) (2 points each) Find the equations of the vertical, horizontal, and oblique asymptotes for each of the following functions. Justify your answers. To justify your vertical and horizontal asymptotes, it is enough to just write down what limit you would have to find to verify that there is indeed a vertical or horizontal asymptote where you say there is one.

a) $\frac{\sin x}{x}$

Horizontal: $\lim_{x \to \pm\infty} \frac{\sin x}{x} = 0 \Rightarrow \boxed{y = 0}$

Vertical: $\lim_{x \to c^{\pm}} \frac{\sin x}{x} \neq \pm\infty$ for any c $\Rightarrow \boxed{\text{no V.A.}}$

b) $\sin \frac{1}{x}$

Horizontal: $\lim_{x \to \pm\infty} \sin \frac{1}{x} = 0 \Rightarrow \boxed{y = 0}$

Vertical: $\lim_{x \to c^{\pm}} \sin \frac{1}{x} \neq \pm\infty$ for any $c \Rightarrow \boxed{\text{no V.A.}}$

c) $\frac{2x^2-3x+1}{x^2+1}$

Horizontal: $\lim_{x \to \pm\infty} \frac{2x^2-3x+1}{x^2+1} = 2 \Rightarrow \boxed{y = 2}$

Vertical: $\lim_{x \to c^{\pm}} \frac{2x^2-3x+1}{x^2+1} \neq \pm\infty$ for any c $\Rightarrow \boxed{\text{no V.A.}}$

d) $\frac{x^3-4}{x^2-2}$

Horizontal: $\lim_{x \to \pm\infty} \frac{x^3-4}{x^2-2} = \pm\infty \Rightarrow \boxed{\text{no H.A.}}$

Vertical: $\lim_{x \to \pm\sqrt{2}^{\pm}} \frac{x^3-4}{x^2-2} = \pm\infty \Rightarrow \boxed{x = \pm\sqrt{2}}$

oblique:
$$\begin{array}{r|l} & x \\ x^2-2 & x^3-4 \\ & -x^3+2x \\ & 2x-4 \end{array}$$
$\rightsquigarrow \boxed{y = x}$

e) $\frac{3x^5-4x}{x^5-1}$

Horizontal: $\lim_{x \to \pm\infty} \frac{3x^5-4x}{x^5-1} = 3 \Rightarrow \boxed{y = 3}$

Vertical: $\lim_{x \to 1^{\pm}} \frac{3x^5-4x}{x^5-1} = \pm\infty \Rightarrow \boxed{x = 1}$