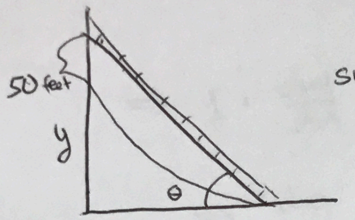
1) (10 points) A 50 foot long ladder is leaning on a flat wall which is perpendicular to the floor, and sliding down. Let θ be the angle that the ladder makes with the floor. This angle θ is changing at a constant rate of -0.2 radians per second. How fast is the ladder slipping down the wall when it is touching the wall 25 feet above the ground?



$\sin\theta = y/50$

$\Rightarrow y = 50\sin\theta.$

$\frac{d\theta}{dt} = -0.2$ rad/sec

$\frac{dy}{dt} = \frac{d}{dt}(50\sin\theta) = 50\cos\theta\frac{d\theta}{dt}$

$= 50(-0.2)\cos\theta$

$= -10\cos\theta.$

when $y = 25$, $\theta = \sin^{-1}(y/50) = \sin^{-1}\left(\frac{25}{50}\right) = \frac{\pi}{6}$

$\Rightarrow \frac{dy}{dt} = -10\left(\underbrace{\cos\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}}\right) = \boxed{-5\sqrt{3}\text{ feet/sec}}$

2) a) (5 points) Let $\sin(x+y) = x^2$. Find $\frac{dy}{dx}$.

differentiate both sides:

Chain Rule: $\cos(x+y)\underbrace{\left(1+\frac{dy}{dx}\right)}_{\frac{d}{dx}(x+y)} = 2x$

$\Rightarrow 1+\frac{dy}{dx} = 2x / \cos(x+y)$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{\cos(x+y)} - 1$

b) (5 points) Again, letting $\sin(x+y) = x^2$, find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{2x}{\cos(x+y)} - 1\right)$$

$$= \frac{2\cos(x+y) - 2x\left(\cos(x+y)\right)'}{\cos^2(x+y)}$$

Quotient Rule

$$= \frac{2\cos(x+y) - 2x(-\sin(x+y))\cdot\left(1+\frac{dy}{dx}\right)}{\cos^2(x+y)}.$$

chain Rule

plug in $\frac{dy}{dx} = \frac{2x}{\cos(x+y)} - 1$ to get complete answer.

3) Use any differentiation rules(sum/product/quotient/chain/etc) or logarithmic differentiation to find derivatives of the following functions. No need to simplify answers.

a) (4 points) $f(x) = \frac{\ln(x+4)}{(x^2+1)(x+3)}$

we use logarithmic differentiation.

$$f'(x) = f(x)\,(\ln(f(x)))' = f(x)\left(\ln(\ln(x+4)) - \ln(x^2+1) - \ln(x+3)\right)'$$

$$= f(x)\left(\frac{1}{(x+4)(\ln(x+4))} - \frac{2x}{x^2+1} - \frac{1}{x+3}\right)$$

Chain rule, $(\ln x)' = \frac{1}{x}$

plug in $f(x) =$

b) (3 points) $f(x) = (x+2)^x$ *Hint: use logarithmic differentiation*

$$f'(x) = f(x)(\ln(f(x)))' = f(x)\cdot(x\ln(x+2))'$$

$$= (x+2)^x\left(\ln(x+2) + \frac{x}{x+2}\right)$$

product, chain rule

$(2.01+2)^{2.01}$

c) (3 points) Use your answer in part (b) to estimate $4.01^{2.01}$. No need to simplify. If you didn't do part (b) explain how part (b) can be used to do this.

$$\frac{\Delta y}{\Delta x} \approx f'(2) \overset{\text{part (b)}}{=} (2+2)^2\left(\ln(2+2) + \frac{2}{2+2}\right)$$

here $\Delta x = 0.01$

$x = 2$

$y = f(x) = (2+2)^2 = 16$

$$= 16\left(\ln 4 + \frac{1}{2}\right)$$

$$= 16\ln 4 + 8$$

$$\Rightarrow \Delta y = (16\ln 4 + 8)\underbrace{\Delta x}_{0.01} = 0.16\ln 4 + 0.08$$

$$\Rightarrow f(2.01) \approx y + \Delta y = \boxed{16 + 0.16\ln 4 + 0.08}$$

4) Compute the following derivatives $f'(x)$ using the LIMIT DEFINITION. Be sure to show all your work.

a) (5 points) $f(x) = x^3 - x$

$$\lim_{h \to 0} \frac{((x+h)^3 - x - h) - (x^3 - x)}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-x} - h \cancel{-x^3} \cancel{+x}}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1)$$

$$= \boxed{3x^2 - 1}$$

plug in h=0
since polynomial
in h

b) (5 points) $f(x) = 9 - \sqrt{2x+3}$

$$\lim_{h\to 0} \frac{9 - \sqrt{2(x+h)+3} - (9 - \sqrt{2x+3})}{h}$$

$$= \lim_{h\to 0} \frac{-\sqrt{2x+2h+3} + \sqrt{2x+3}}{h} \cdot \frac{\sqrt{2x+2h+3} + \sqrt{2x+3}}{\sqrt{2x+2h+3} + \sqrt{2x+3}}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} \cancel{+3} - \cancel{2x} - 2\cancel{h} \cancel{-3}}{\cancel{h}(\sqrt{2x+2h+3} + \sqrt{2x+3})}$$

$$= \frac{-2}{\sqrt{2x+3} + \sqrt{2x+3}} = \boxed{\frac{-1}{\sqrt{2x+3}}}$$

plug in
h=0 since
~~polynomial~~
polynomial

5) a) (4 points) Compute the derivative of $f(x) = x^2 \sin^{-1} x$. At each step in your solution, clearly explain what differentiation rules you're using.

$$(x^2 \sin^{-1} x)' = (x^2)' \sin^{-1}(x) + x^2 (\sin^{-1} x)'$$

product rule

$$= \boxed{2x \sin^{-1} x + \frac{x^2}{\sqrt{1-x^2}}}$$

power rule, derivative of $\sin^{-1} x$

b) (4 points) Compute the derivative of $f(x) = x(\tan^{-1} x)^2$. At each step in your solution, clearly explain what differentiation rules you're using.

$$(x (\tan^{-1}(x))^2)' = (\tan^{-1} x)^2 + x ((\tan^{-1} x)^2)'$$

product rule

$$= \boxed{(\tan^{-1} x)^2 + x \cdot 2 \tan^{-1} x \cdot \frac{1}{1+x^2}}$$

chain rule, derivative of $\tan^{-1} x$

c) (2 points) Let $f(x) = x^3 - 2$ and let $g(x)$ be the inverse of $f(x)$. Compute $g'(6)$.

Note $g(6) = 2$ since $f(2) = 8 - 2 = 6$.

$$g'(6) = \frac{1}{f'(g(6))} = \frac{1}{3 \cdot 2^2} = \boxed{\frac{1}{12}}$$

$f'(x) = 3x^2$