Math 250A Homework 1, due 10/1/2021

1) Let G be a group. A *commutator* in G is an element of the form $aba^{-1}b^{-1}$ with $a,b \in G$. Let G^c be the subgroup generated by the commutators, called the *commutator subgroup*. Show that G^c is normal in G. *Challenge:* Show that any homomorphism ϕ of G into an abelian group factors through G/G^c, meaning that there exists a map f such that $\phi = f \circ \pi$ where $\pi : G/G^c$ is the canonical morphism.

2) Two subgroups H and H' of a group G are said to be *commensurable* if $H \cap H'$ is of finite index in both H and H'. Show that commensurability is an equivalence relation on the subgroups of G.

3) a) Let G be a finite group and let $H \leq G$. Given $g \in G$, does $gHg^{-1} \subset H$ imply $g^{-1}Hg \subset H$?

b) Let G be an infinite group and let $H \leq G$. Given $g \in G$, does $gHg^{-1} \subset H$ imply $g^{-1}Hg \subset H$? *Hint: consider $G = GL_2(\mathbb{Q})$ and let*

$$H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \textit{ where } n \in \mathbb{Z} \right\}$$

4) a) Let H, N be normal subgroups of a finite group G. Assume that the orders of H, N are relatively prime. Prove that $xy = yx$ for all $x \in H$ and $y \in N$, and that $H \times N \cong HN$.

b) Let $H_1, \ldots, H_r$ be normal subgroups of G such that the order of H_i is relatively prime to the order of H_j for $i \neq j$. Prove that

$$H_1 \times \ldots \times H_r \cong H_1 \cdots H_r$$

5) Let p be a prime and let G be of order p^n. Such a group is called a p-group, and it is known that for any nontrivial p-group G, the center $Z(G) \neq \mathbf{1}$. Show that G has a chain of subgroups

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \mathbf{1}$$

such that G_i is normal in G and $[G : G_i] = p^i$ for all i. What are the composition factors of G?

Hint: Use the fact that $Z(G) \neq \mathbf{1}$ to produce an element $x \in Z(G)$ of order p. Prove by induction, considering the quotient group $G/\langle x \rangle$.

6) The dihedral group D_8 containing 8 elements has seven different composition series. Find all of them.

7) a) Show that an abelian group has a composition series if and only if it is finite.

b) Let F be a field and let $\mathrm{GL}_n(F)$ denote the group of $n \times n$ invertible matrices with entries in F (the group operation is matrix multiplication). Show that $\mathrm{GL}_n(F)$ has a composition series if and only if F is finite.