Math 250A Homework 2, due 10/8/2021

1) Prove that the group of inner automorphisms of a group G is normal in $\mathrm{Aut}(G)$.

2) Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

3) Let P be a p-group. Let A be a normal subgroup of order p. Prove that A is contained in the center of P. In case we haven't gotten to this in class yet, given a prime p, a group of order p^n for some $n \geq 0$ is called a p-group.

4) A nontrivial fact is that $[\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)] = 2$. In this problem we will show that $[\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)] \leq 2$ and that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$ for all $n \geq 3$ and $n \neq 6$. Throughout this problem, $n \geq 3$.

a) Let C be the conjugacy class of any transposition in S_n, and let C' be the conjugacy class of any element of order 2 in S_n which is not a transposition. Show that $|C| \neq |C'|$ unless $n = 6$ and C' is the conjugacy class of a product of three disjoint transpositions. Deduce that $\mathrm{Aut}(S_6)$ has a subgroup of index at most 2 which sends transpositions to transpositions, and that any automorphism of S_n where $n \neq 6$ sends transpositions to transpositions.

b) Show that any automorphism that sends transpositions to transpositions is inner, and deduce that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$ if $n \neq 6$ and $[\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)] \leq 2$ from part (a).

5) Show that a simple group whose order is $\geq r!$ cannot have a proper nontrivial subgroup of index r.

6) In this problem, you are allowed to assume that A_n is simple for $n \geq 5$. Show that S_n has no proper subgroups of index $< n$ other than A_n for $n \geq 5$.

7) A *chief series* of a group G is a series of subgroups

$$G = G_0 > G_1 > \cdots > G_r = 0$$

such that $G_i \lhd G$ for all $1 \leq i \leq r$ and such that no normal subgroup of G is contained properly between any two terms in the series. The factors G_i/G_{i+1} in this series are called *chief factors*.

a) We say that N is a *minimal normal subgroup* of a group G if $1 \neq N \trianglelefteq G$ such that no nontrivial normal subgroup of G is properly contained in N. Show that every finite group G has a chief series and that any minimal normal subgroup N of a finite group G is a chief factor in some chief series of G. (In fact, chief factors, much like composition factors, are "unique" to a group which admits a chief series, and if you'd like you can formulate and prove for yourself the analog of Jordan-Hölder for chief series)

b) Show that any group having a composition series also has a chief series.

8) A *central series* of a group G is a series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_r = 0$$

such that $G_i \trianglelefteq G$ for all $1 \leq i \leq r$ and such that each quotient G_i/G_{i+1} is contained in the center of G/G_{i+1}. A group G is called *nilpotent* if it admits a central series.

a) Show that nilpotent groups are solvable.

b) Give an example of a solvable group which is not nilpotent.