Math 250A Homework 3, due 10/22/2021

1) Let G be a finite group and H a subgroup. Let P_H be a p-Sylow subgroup of H. Prove that there is a p-Sylow subgroup P of G such that $P_H = P \cap H$.

2) Let G be a group of order p^3 where p is prime, and G is not abelian. Let Z be its center. Let C be a cyclic group of order p.

a) Show that $Z \cong C$ and $G/Z \cong C \times C$.

b) Show that every subgroup of G of order p^2 contains Z and is normal.

c) Suppose $x^p = 1$ for all x in G. Show that G contains a normal subgroup $H \cong C \times C$.

3) a) Prove that one of the Sylow subgroups of a group of order 40 is normal.

b) Prove that one of the Sylow subgroups of a group of order 12 is normal.

4) Let G be a finite group, and let r be the number of conjugacy classes of G. Show that

$$|\{(a,b) \in G \times G \mid ab = ba\}| = r|G|.$$

5) Let G be a finite group, let $N \trianglelefteq G$, and let P be a Sylow subgroup of N. Show that $G = N_G(P)N$, where $N_G(P)$ denotes the normalizer of P in G.

6) a) Let G be a group such that p, q are two distinct prime divisors of $|G|$. Suppose that P is the only p-Sylow subgroup of G and Q is the only q-Sylow subgroup of G. Show that the elements of P commute with the elements of Q.

b) Show that any group of order 45 is abelian.

7) Show that every Sylow subgroup is normal in G if and only if G is the direct product of its Sylow subgroups.

8) Let H be a cyclic group and let N be an arbitrary group. If ϕ and ψ are injective homomorphisms from H to $\mathrm{Aut}(N)$ such that $\phi(H) = \psi(H)$, show that $N \rtimes_\phi H \cong N \rtimes_\psi H$.