Math 250A Homework 4, due 10/29/2021

1) Show that if G is a finite group which has only one subgroup of order d for every $d \mid |G|$ then G is cyclic. *(Hint: use a Sylow theorem to reduce this problem to the case where G is a p-group first)*

2) a) Let $N \lhd G$ where G is a finite group and let p be a prime dividing $|N|$. Show that for any p-Sylow subgroup P of G we have that $P \cap N$ is a p-Sylow subgroup of N and that all p-Sylow subgroups of N arise in this way. Thus the number of p-Sylow subgroups of N is at most the number of p-Sylow subgroups of G.

b) Give a counterexample to the statement in (a) with $N \lhd G$ replaced by $N < G$.

3) a) Show that $\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5) \cong \mathrm{GL}_2(\mathbb{F}_5)$ and that $|\mathrm{GL}_2(\mathbb{F}_5)| = 480$.

b) Give an example of a nonabelian group of order 75.

c) For this problem, it might be helpful to know the following:

Let K be a cyclic group and let H be an arbitrary group. Let $\phi : K \to Aut(H)$ and $\psi : K \to Aut(H)$ be injective homomorphisms such that $\psi(K)$ is conjugate to $\phi(K)$ in $Aut(H)$. Then $H \rtimes_\phi K \cong H \rtimes_\psi K$.

With this in mind, classify all groups of order 75.

4) Show that a finite group (not necessarily commutative) is cyclic if, for each $n > 0$, it contains at most n elements of order dividing n.

5) Find a primitive element for the field $\mathbb{Q}(\sqrt{3}, \sqrt{7})$ over $\mathbb{Q}$, i.e., an element α such that $\mathbb{Q}(\sqrt{3}, \sqrt{7}) = \mathbb{Q}(\alpha)$.

6) Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$ and let $m(x)$ denote its minimal polynomial over $\mathbb{Q}$. Suppose $\beta \in \mathbb{C}$ is some other root of $m(x)$. Show that the map $\sigma : \mathbb{Q}(\alpha) \to \mathbb{C}$ which for every $f \in \mathbb{Q}[x]$ takes $f(\alpha)$ to $f(\beta)$ is a well defined field homomorphism such that $\sigma(x) = x$ for all $x \in \mathbb{Q}$.