Math 250A Homework 5, due 11/12/2021

1) Find the degree of the splitting field of x^6+1 over $\mathbb{Q}$. How about over $\mathbb{F}_2$?

2) a) Let F be a field, let $f(x) \in F[x]$ be a polynomial of prime degree. Suppose for every field extension K of F that if f has a root in K, then f splits over K. Prove that either f is irreducible over F or f has a root (and hence splits) in F.

b) Give two examples of situations in which the hypotheses in (a) hold for characteristic $p > 0$ fields, and one example of a situation in which the hypotheses in (a) hold in a characteristic 0 field.

3) Let F be a field. Show that the rational function field $F(x)$ is not algebraically closed.

4) Let F be a finite extension of $\mathbb{Q}$. Show that F is not algebraically closed.

5) Let F be a field of characteristic p.

a) Let $F^p = \{a^p \mid a \in F\}$. Show that F^p is a subfield of F.

b) If $F = \mathbb{F}_p(x)$ is the rational function field in one variable over $\mathbb{F}_p$, determine $[F : F^p]$.

6) Show that every element of a finite field is a sum of two squares.

7) Let $f(x)$ be an irreducible polynomial over F of degree n and let K be a field extension of F with $[K : F] = m$. If $\gcd(n,m) = 1$, show that f is irreducible over K.

8) Let K and L be extensions of F. Show that KL is normal over F if both K and L are normal over F. Is the converse true?