Math 250A Homework 6, due 11/24/2021

1) Let K/F be a Galois extension with $[K : F] = n$. Show that if $p|n$ is a prime then there is a subfield L of K with $[K : L] = p$.

2) Let K/F be a Galois extension with $\mathrm{Gal}(K/F) \cong A_4$. Show that there is no intermediate field M of the extension K/F such that $[M : F] = 2$.

3) Show that if K/F is a Galois extension such that there are no proper intermediate fields between K and F, then $[K : F]$ is a prime number. Is this still true if K/F is not a Galois extension?

4) Let K/F be a Galois extension and $\alpha \in K$ and $H = \mathrm{Gal}(K/F(\alpha))$. Let $[K : F] = n$ and $[F(\alpha) : F] = r$. Suppose that $\{\tau_1, \cdots, \tau_r\}$ is a set of left coset representatives of H in $\mathrm{Gal}(K/F)$. Show that the minimal polynomial of α over F is given by

$$m(x) = \prod_{i=1}^{r}(x - \tau_i(\alpha))$$

and show that

$$\prod_{\sigma \in \mathrm{Gal}(K/F)} (x - \sigma(\alpha)) = m(x)^{n/r}$$

5) Let K/F be a Galois extension with $\mathrm{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{char}(F) \neq 2$. Show that $K = F(\sqrt{\alpha}, \sqrt{\beta})$ for some $\alpha, \beta \in F$.

6) Let K be the splitting field of $x^8 - 1$ over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$ and describe all intermediate fields of $K/\mathbb{Q}$.

7) Let $S = \{\sqrt{p} \mid p \text{ a prime}\}$ and $K = \mathbb{Q}(S)$. For $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$ define

$$Y_\sigma = \{\sqrt{p} \mid \sigma(\sqrt{p}) = -\sqrt{p}\}$$

Show the following:

(i) K is normal and separable over $\mathbb{Q}$

(ii) If $Y_\sigma = Y_\tau$, then $\sigma = \tau$

(iii) If Y is a subset of S then there exists $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$ such that $Y = Y_\sigma$

(iv) Let $P(S)$ denote the set of all subsets of S. Show that

$$[K : \mathbb{Q}] = |S|$$

and

$$|\mathrm{Aut}(K/\mathbb{Q})| = |P(S)|$$

[Note that it follows that $|\mathrm{Aut}(K/\mathbb{Q})| > [K : \mathbb{Q}]$]

8) Let K be a subfield of $\mathbb{C}$ such that $K/\mathbb{Q}$ is a Galois extension. Let $c \in \mathrm{Aut}(\mathbb{C})$ be complex conjugation.

a) Show that $c(K) = K$ and the restriction $c|_K$ of c to K is an element of $\mathrm{Gal}(K/\mathbb{Q})$.

b) Show that $\mathcal{F}(c|_K) = K \cap \mathbb{R}$ and $[K : K \cap \mathbb{R}] \leq 2$.

c) Give an example of K where $[K : K \cap \mathbb{R}] = 1$ and an example of K where $[K : K \cap \mathbb{R}] = 2$.

9) Let k be a field of characteristic $p > 0$, let $K = k(x, y)$ be the rational function field in two variables over k, and let $F = k(x^p, y^p)$.

a) Prove that $[K : F] = p^2$.

b) Prove that $K^p \subseteq F$ (see Homework 5 for definition of K^p).

c) Prove that there is no $\alpha \in K$ with $K = F(\alpha)$.

d) Exhibit an infinite number of intermediate fields of K/F.

10) Let $K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ and let $F = \mathbb{Q}$. Show directly that there exists a $\sigma \in \mathrm{Aut}(K/F)$ such that $\sigma(\sqrt[3]{2}) = \zeta_3 \sqrt[3]{2}$ and $\sigma(\zeta_3) = \zeta_3^2$.

11) Let K/F be a Galois extension. Two intermediate fields L_1, L_2 are called conjugate if there is $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(L_1) = L_2$. Characterize conjugate intermediate fields in terms of the corresponding subgroups of $\mathrm{Gal}(K/F)$.

12) Consider the quaternion group Q_8 which as a set is given by $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication determined by $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$. Let F be a field. Find a degree 4 polynomial $f(x) \in F[x]$ whose splitting field over F is Galois with Galois group isomorphic to Q_8 or show that no such polynomial exists.