Math 250A Homework 7, due 12/3/2021

In the first few problems, we will walk through the proof of the fundamental theorem of algebra. First, a definition:

Let K/F be a finite extension. The **normal closure** *of K/F is the splitting field over F of*

$$\{\textit{minimal polynomial of } \alpha \textit{ over } F \mid \alpha \in K\}$$

1. Let K/F be a finite extension and let N be the normal closure of K/F.

a) Show that N is a normal extension of F containing K and whenever M is a normal extension of F with $K \subseteq M \subseteq N$ then $M = N$.

b) Show that if $K = F(\alpha_1, \ldots, \alpha_r)$ then N is the splitting field of the set of minimal polynomials of α_i's over F.

c) Show that N/F is a finite extension.

d) Show that if K/F is separable then N/F is Galois.

e) Pick a favorite finite extension of some field F that is not normal and find its normal closure.

2. a) Show that if $K/\mathbb{R}$ is a finite extension with odd degree, then $K = \mathbb{R}$.

b) Show that if $K/\mathbb{C}$ is a finite extension such that $[K : \mathbb{C}] \leq 2$ then $K = \mathbb{C}$.

3. Let $K/\mathbb{C}$ be a finite extension. Suppose N is the normal closure of $K/\mathbb{R}$. Suppose $N \neq \mathbb{C}$.

a) Given the above, show 2 divides $|\mathrm{Gal}(N/\mathbb{R})|$ and that there is no nontrivial extension of odd degree $L/\mathbb{R}$ contained in N.

b) Show that both $\mathrm{Gal}(N/\mathbb{R})$ and $\mathrm{Gal}(N/\mathbb{C})$ are nontrivial 2-groups.

c) Show that $\mathrm{Gal}(N/\mathbb{C})$ has an index 2 subgroup H and determine the degree of $\mathcal{F}(H)/\mathbb{C}$. Conclude that $\mathbb{C}$ is algebraically closed.

4. Determine the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. Find the intermediate fields and corresponding subgroups.

5. Let K/F be a finite dimensional extension. It is called simple if there is $\alpha \in K$ such that $K = F(\alpha)$.

(1) Show that K/F is simple if and only if there are finitely many intermediate fields of K/F.
(2) Show that if K/F is separable and finite dimensional then it is simple.
(3) Show that there exists a finite dimensional extension K/F which is not simple.

Some extra problems for your enjoyment (OPTIONAL)

6. Let K be the splitting field of $x^3 - 3x + 1$ over $\mathbb{Q}$.

(1) Show that $[K : \mathbb{Q}] = 3$.
(2) Show that $\mathrm{Gal}(K/\mathbb{Q})$ is solvable but K is not a radical extension of $\mathbb{Q}$.

In the following problems we discuss some aspects of Galois theory for infinite dimensional extensions. Some notation: We now *define* an extension K/F to be Galois iff it is the splitting field of a collection of separable polynomials (this agrees with our usual definition in the finite dimensional case). We proved a result in class that implies that K/F is Galois iff $\mathcal{F}(\mathrm{Gal}(K/F)) = F$ even in the case of infinite dimensional extensions.

Let $\mathcal{F}$ denote the collection of intermediate fields of K/F which are finite dimensional Galois extensions of F and let $\mathcal{G}$ be the collection of subgroups of $\mathrm{Gal}(K/F)$ which are given by $\mathrm{Gal}(K/E)$ for some $E \in \mathcal{F}$.

7. Show: if $\alpha_1, \cdots, \alpha_n \in K$ then there is $E \in \mathcal{F}$ with $\alpha_i \in E$ for all i.

8. Let $N = \mathrm{Gal}(K/E)$ for $E \in \mathcal{F}$. Show that $E = \mathcal{F}(N)$ and N is a normal subgroup of $\mathrm{Gal}(K/F)$ and $\mathrm{Gal}(K/F)/N \cong \mathrm{Gal}(E/F)$.

9. Show:

(1) $\bigcap_{N \in \mathcal{G}} N = \{id\}$
(2) if $N_1, N_2 \in \mathcal{G}$ then $N_1 \cap N_2 \in \mathcal{G}$

You might have seen the definition of a topological space in some other course, but if not, here is the definition: A topology on a set S is a collection $\mathcal{T}$ of subsets of X such that

$S, \varnothing \in \mathcal{T}$, if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$, any union of sets in $\mathcal{T}$ is in $\mathcal{T}$ again. A topological space is a set together with a topology on it and the sets in $\mathcal{T}$ are called open sets and the subsets of S which are complements of an element in $\mathcal{T}$ are called closed sets.

For example one can take $S = \mathbb{C}$ and define a topology on $\mathcal{T}$ by defining a subset $U \subseteq \mathbb{C}$ to be open if whenever $x \in U$ then there is $\epsilon > 0$ such that the open disc centered at x of radius ϵ is contained in U. The topologies that arise naturally in algebra are often of a more unintuitive nature.

10. Let K/F be a (possibly infinite dimensional) Galois extension.

(1) Show that the following defines a topology on $\mathrm{Gal}(K/F)$: A subset U of $\mathrm{Gal}(K/F)$ is open if either $U = \varnothing$ or if it is a union of sets of the form σN where $N \in \mathcal{G}$ and $\sigma \in \mathrm{Gal}(K/F)$. (This topology is called the Krull topology)

(2) Show that a set of the form σN for $N \in \mathcal{G}$ and $\sigma \in \mathrm{Gal}(K/F)$ is both open and closed. (Such a set is also called clopen. It might be instructive to compare the existence of these clopen sets in the Krull topology on $\mathrm{Gal}(K/F)$ to the more "geometric" topology on $\mathbb{C}$ described earlier.)

11. Let S be a set with a topology $\mathcal{T}$ on it. The closure $\overline{A}$ of a subset A of S is defined to be the intersection of all closed subsets of S that contain A. Suppose that $x \in S$ satisfies the following: For any open set U with $x \in U$ one has $U \cap A \neq \varnothing$. Show that $x \in \overline{A}$.

12. Let K/F be a (possibly infinite dimensional) Galois extension and let H be a subgroup of $\mathrm{Gal}(K/F)$.

(1) Show that $\mathrm{Gal}(K/\mathcal{F}(H))$ is a closed subgroup of $\mathrm{Gal}(K/F)$.

(2) Show that $\mathrm{Gal}(K/\mathcal{F}(H))$ is contained in the closure of H in $\mathrm{Gal}(K/F)$ with respect to the Krull topology. [Hint: Use previous exercise] Deduce that $\mathrm{Gal}(K/\mathcal{F}(H))$ is the closure of H in $\mathrm{Gal}(K/F)$.

13. Let K/F be a (possibly infinite dimensional) Galois extension. Show the fundamental theorem of infinite Galois theory: There is an inclusion reversing 1-to-1 correspondence between intermediate fields of K/F and closed subgroups of $\mathrm{Gal}(K/F)$ with respect to the Krull topology.