Math 250A Final Exam Prep

1. Let G be a group of order 24 which contains a non-normal subgroup H of order 8.
   (a) Including H, how many conjugates of H are there in G?
   (b) Using the conjugates of H from the previous part, define a homomorphism of G into the group S_3.
   (c) Conclude that G is not simple.

2. In this problem, you may use part (a) to prove part (b) and part (b) to prove part (c) even if you fail to prove part (a) and/or (b).

   Let G be a finite solvable group. A subgroup H of G is called characteristic in G if it is invariant under any automorphism of G.

   (a) If N is a nontrivial minimal normal subgroup of G, meaning that it is nontrivial and contains no proper nontrivial subgroup which is normal in G, show that N is abelian.
   (b) Show that if N is a nontrivial minimal normal subgroup of G then $P := \{x \in N \mid x^p = 1\}$ where $p \,|\, |N|$ is prime is a characteristic subgroup of N. Deduce that N is a p-group.
   (c) Let M be a maximal proper subgroup of G. Show that $[G : M]$ is a power of a prime. (Hint: Let N be a nontrivial minimal normal subgroup of G; consider the case $N \subset M$ and the case $N \not\subset M$).

3. Let t be an indeterminate and consider the function field $K = \mathbb{C}(t)$. Consider the field extension K/F where $F = \mathbb{C}(t^4)$.
   (a) Prove that K is the splitting field over F of $f(x) = x^4 - t^4 \in F[x]$ and show that K/F is Galois.
   (b) Prove that $f(x) = x^4 - t^4$ is irreducible in $F[x]$.
   (c) Prove that $\mathrm{Gal}(K/F) \cong \mathbb{Z}/4\mathbb{Z}$ and for each $\sigma \in \mathrm{Gal}(K/F)$ write down explicitly $\sigma(t)$. Write down a generator of $\mathrm{Gal}(K/F)$.
   (d) Determine all the subgroups of $\mathrm{Gal}(K/F)$ and the corresponding intermediate fields of K/F under the Galois correspondence.

4. Let K be a finite field with an algebraic closure $\overline{K}$ such that $\mathrm{char}(K) = p > 0$. Let $a \in K$ and consider
$$f(x) = x^p - x + a \in K[x]$$
   (a) Let $\alpha \in \overline{K}$ be a root of $f(x)$. Show that $\alpha + 1$ is also a root of $f(x)$.
   (b) Show that either $f(x)$ is irreducible in $K[x]$ or $f(x)$ has all its roots in K.
   (c) Let $a \in \mathbb{F}_p$ be non-zero. Show that the splitting field of $x^p - x + a$ over $\mathbb{F}_p$ is an extension of degree p of $\mathbb{F}_p$.
   (d) Show that the polynomial $x^p - x + n$ is irreducible in $\mathbb{Q}[x]$ for an infinite number of choices of $n \in \mathbb{Z}$.

5. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

6. Let $d > 0$ be a square-free integer. Show that $\mathbb{Q}(\sqrt[8]{d}, i)/\mathbb{Q}(\sqrt{d})$ is Galois and that its Galois group is the dihedral group with 8 elements. Choose four intermediate fields of this extension and determine the Galois groups of $\mathbb{Q}(\sqrt[8]{d}, i)$ over these fields.