ORBITS ON K3 SURFACES OF MARKOFF TYPE

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Abstract. Let $\mathcal{W} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a surface given by the vanishing of a $(2, 2, 2)$-form. These surfaces admit three involutions coming from the three projections $\mathcal{W} \to \mathbb{P}^1 \times \mathbb{P}^1$, so we call them *tri-involutive K3* (TIK3) *surfaces*. By analogy with the classical Markoff equation, we say that $\mathcal{W}$ is of *Markoff type* (MK3) if it is symmetric in its three coordinates and invariant under double sign changes. An MK3 surface admits a group of automorphisms $\mathcal{G}$ generated by the three involutions, coordinate permutations, and sign changes. In this paper we study the $\mathcal{G}$-orbit structure of points on TIK3 and MK3 surfaces. Over finite fields, we study fibral connectivity and the existence of large orbits, analogous to work of Bourgain, Gamburd, Sarnak and others for the classical Markoff equation. For a particular 1-parameter family of MK3 surfaces $\mathcal{W}_k$, we compute the full $\mathcal{G}$-orbit structure of $\mathcal{W}_k(\mathbb{F}_p)$ for all primes $p \le 79$, and we use this data as a guide to find many finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$, including a family of orbits of size 288 parameterized by a curve of genus 9.

Contents

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1. Introduction

The classical Markoff equation is the affine surface

$$\mathcal{M} : x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

It admits three involutions coming from the three projections $\mathcal{M} \to \mathbb{A}^2$, and these three involutions, together with double sign changes and coordinate permutations, generate the automorphism group $\mathcal{G}_{\mathcal{M}} := \operatorname{Aut}(\mathcal{M})$ of $\mathcal{M}$. A classical theorem of Markoff [22] says that the set of integer solutions $\mathcal{M}(\mathbb{Z})$ consists of two orbits, one "small" $\mathcal{G}_{\mathcal{M}}$-orbit containing the single point $(0, 0, 0)$, and one "large" $\mathcal{G}_{\mathcal{M}}$-orbit containing $(1, 1, 1)$.

The orbit structure of $\mathcal{M}(\mathbb{F}_p)$ under the action of $\mathcal{G}_{\mathcal{M}}$ has been studied by a number of authors, including Baragar [1] and Bourgain–Gambard–Sarnak [8]. The latter prove that for most primes p, there is only one large orbit in $\mathcal{M}(\mathbb{F}_p)$. The proof is an ingenious algorithm that jumps between differently oriented fibers, using the Hasse–Weil estimate to say that if a point on a "vertical" fiber has a large enough orbit, then one of the "horizontal" orbits consists of an entire "horizontal" fiber. The proof implicitly relies on the fact that each fiber of $\mathcal{M}$ is a torus and that the fibral automorphisms are toral translations (i.e., $\mathbb{G}_m$-translations), which in [8] are called rotations. See Section 2 for more details.

The first goal of this paper is to study similar questions on an analogous family of projective surfaces that admit three involutions. We define the family of *tri-involutive K3* (TIK3) *surfaces* to be the hypersurfaces

$$\mathcal{W} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \tag{2}$$

given by the vanishing of a $(2, 2, 2)$-form. These surfaces have three involutions $\sigma_1, \sigma_2, \sigma_3$ coming from the three projections $\mathcal{W} \to \mathbb{P}^1 \times \mathbb{P}^1$. The study of the geometry and arithmetic of these surfaces is of course not new; see Section 5 for a brief history. In this paper we study the fibral structure of $\mathcal{W}(\mathbb{F}_p)$ for the three projections and the orbit structure of $\mathcal{W}(\mathbb{F}_p)$ under the action of $\operatorname{Aut}(\mathcal{W})$. For example, we prove the following fibral linking result, which is a TIK3 analogue of [8, Proposition 6] for the Markoff equation. See Theorem 6.5 for further details and a proof.

Theorem 1.1. *Assume that* $p > 100$*, and let* $\mathcal{W}/\mathbb{F}_p$ *be a TIK3 surface. Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be fibers of* $\mathcal{W}(\mathbb{F}_p) \to (\mathbb{P}^1)^2(\mathbb{F}_p)$ *for any two of the projections. Then there is a fiber* $\mathcal{F}_3$ *for one of the projections satisfying*

$$\mathcal{F}_1 \cap \mathcal{F}_3 \neq \emptyset \quad \textit{and} \quad \mathcal{F}_2 \cap \mathcal{F}_3 \neq \emptyset.$$

Our second goal is inspired by the classification of finite orbits on Markoff-type surfaces over $\mathbb{C}$. For example, the papers [11, 17, 21] contain a detailed description of the $(a, b, c, d) \in \mathbb{C}$ for which the surface

$$\mathcal{M}_{a,b,c,d} : x^2 + y^2 + z^2 + ax + by + cz + dxyz = 0.$$

has one or more finite orbits. The existence of such orbits turns out to be related to algebraic solutions to Painlevé differential equations. It is likewise true [10] that a (non-degenerate) TIK3 surface $\mathcal{W}(\mathbb{C})$ has only finitely many finite orbits, but the methods used to classify the orbits for Markoff-type equations do not seem easily applicable to the TIK3 situation.

Generically, the automorphism group of $\mathcal{W}$ is generated by the three automorphisms. Since the Markoff equation (1) admits additional automorphisms, we consider an analogous family of TIK3 surfaces, which we call *Markoff-type K3* (MK3) *surfaces*. These are the TIK3 surfaces (2) that are invariant under coordinate permutations and double sign changes. See Proposition 7.5 for a description of the full 4-dimensional family of MK3 surfaces.

A typical example, which we use as a prototype, is the following one-parameter family of MK3-surfaces $\mathcal{W}_k$. For non-zero k, we define $\mathcal{W}_k$ to be the projective closure in $(\mathbb{P}^1)^3$ of the affine surface

$$\mathcal{W}_k : x^2 + y^2 + z^2 + x^2y^2z^2 + kxyz = 0. \tag{3}$$

In order to understand the orbit structure in $\mathcal{W}_k(\mathbb{F}_p)$, we computed all orbits for $p \leq 79$ and all $k \in \mathbb{F}_p^*$; see Tables 5–8 in Section 11. We use these computations for two purposes.

First, by studying small orbit sizes that appear in $\mathcal{W}_k(\mathbb{F}_p)$ for many different p and k, we find patterns which we use to construct finite orbits in $\mathcal{W}_k(\mathbb{C})$. A full description of our findings is contained in Section 10; see especially Table 3. We illustrate by stating a few results, including some fairly large finite orbits that occur in 1-parameter families:

Proposition 1.2. *Let* $\mathcal{W}_k$ *be the projective closure in* $(\mathbb{P}^1)^3$ *of the affine surface* (3).

- $\mathcal{W}_{-4}(\mathbb{Q})$ *contains an orbit of size* 4 *and* $\mathcal{W}_4(\mathbb{Q})$ *contains an orbit of size* 12.

- *$\mathcal{W}_k\bigl(\mathbb{Q}(i)\bigr)$ contains an orbit of size 48 for every $k \in \mathbb{Q}(i)$.*
- *There is a field $K/\mathbb{Q}$ of degree 8 and an element $k \in K$ so that $\mathcal{W}_k(K)$ has an orbit of size 144.*
- *There is a field $K/\mathbb{Q}$ of degree 8 and an element $k \in K$ so that $\mathcal{W}_k(K)$ has an orbit of size 160.*
- *There is a $k(t) \in \mathbb{Q}(t)$ so that $\mathcal{W}_{k(t)}\bigl(\mathbb{Q}(t)\bigr)$ has an orbit of size 24.*
- *There is a $k(t) \in \mathbb{Q}(i,t)$ so that $\mathcal{W}_{k(t)}\bigl(\mathbb{Q}(i,t)\bigr)$ has an orbit of size 96.*
- *There is an irreducible curve $C/\mathbb{Q}$ of genus 9 and an element $k \in \mathbb{Q}(C)$ in the function field of C so that $\mathcal{W}_k\bigl(\mathbb{Q}(C)\bigr)$ has an orbit of size 288.*

In the spirit of the many uniform boundedness theorems and conjectures in arithmetic geometry and arithmetic dynamics, we pose the following question:

Question 1.3. Does there exist a constant N so that

$$\#\{P \in \mathcal{W}_k(\mathbb{C}) : \text{the orbit of } P \text{ is finite}\} \le N \quad \text{for all } k \in \mathbb{C}^*?$$

More generally, does there exist a constant N so that for every non-degenerate[1] TIK3 surface $\mathcal{W}$ we have

$$\#\{P \in \mathcal{W}(\mathbb{C}) : \text{the } \langle\sigma_1,\sigma_2,\sigma_3\rangle\text{-orbit of } P \text{ is finite}\} \le N?$$

See Question 10.1 for a further discussion of uniform boundedness of finite orbits.

Second, we investigate large orbits in $\mathcal{W}_k(\mathbb{F}_p)$ to see if the methods employed in [8] for the Markoff equation are potentially applicable to the MK3 setting. The fiber-to-fiber jumping strategy employed by [8] uses the fact, which they prove, that if a vertical fibral orbit is sufficiently large, then at least one of the points in that vertical orbit has a horizontal orbit that consists of the entire horizontal fiber. (See Section 4 and Remark 4.4 for further details.) We are interested in the question of whether such a fiber-to-fiber jumping strategy will work on the MK3-surface $\mathcal{W}_k(\mathbb{F}_p)$. In Section 12 we show that the surface $\mathcal{W}_1(\mathbb{F}_{53})$ has a connected component of size 3456, but that the fiber-to-fiber jumping strategy cannot be used to prove that this component is connected. This suggests that additional ideas may be needed to prove the existence of a large orbit in $\mathcal{W}_k(\mathbb{F}_p)$.

[1] See Definition 3.1, but briefly, non-degeneracy means that the three involutions are well-defined.

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2. A brief survey of related work on the Markoff equation

Definition 2.1. Let $a \in K^*$ and $k \in K$. The associated *Markoff equation* is

$$\mathcal{M}_{a,k} : x^2 + y^2 + z^2 = axyz + k. \tag{4}$$

Theorem 2.2. (a) (Markoff [22])

$$\mathcal{M}_{3,0}(\mathbb{Z}) = \bigl\{(0,0,0)\bigr\} \cup \mathcal{G} \cdot (1,1,1).$$

(b) *More generally, for all $a, k \in \mathbb{Z}$ with $a \neq 0$, there is a finite set of points $P_1, \ldots, P_r \in \mathcal{M}_{a,k}(\mathbb{Z})$ such that*

$$\mathcal{M}_{a,k}(\mathbb{Z}) = \bigcup_{i=1}^{r} \mathcal{G} \cdot P.$$

Conjecture 2.3. (Baragar [1, Section V.3], Bourgain–Gambard–Sarnak [7, 8]) *For all primes $p \geq 5$ we have*

$$\mathcal{M}^*_{3,0}(\mathbb{F}_p) = \bigl\{(0,0,0)\bigr\} \cup \bigl(\mathcal{G} \cdot (1,1,1)\bigr).$$

Bourgain–Gambard–Sarnak have a number of deep results related to Conjecture 2.3, including the following:

Theorem 2.4. (a) [8, Theorem 1]

$$\#\mathcal{M}^*_{3,0}(\mathbb{F}_p) \smallsetminus \bigl(\mathcal{G} \cdot (1,1,1)\bigr) = p^{o(1)}, \quad \text{as } p \to \infty.$$

(b) [8, Theorem 2] *Conjecture 2.3 holds for all but possibly $X^{o(1)}$ primes $p \leq X$, as $X \to \infty$.*

Other recent notable results include the following:

- Konyagin–Makarychev–Shparlinski–Vyugin [20] improves Theorem 2.4:
$$\#\mathcal{M}^*_{3,0}(\mathbb{F}_p) \smallsetminus \bigl(\mathcal{G} \cdot (1,1,1)\bigr) \leq \exp\left((\log p)^{2/3+o(1)}\right).$$
- Given a pseudo-Anosov element $g \in \mathrm{Out}(\mathbf{F}_2)$, g induces a permutation g_p on $\mathcal{M}_{1,k}(\mathbb{F}_p)$ for each prime p. Cerbu–Gunther–Magee–Peilen [12] prove that asymptotically, the action of g_p on $\mathcal{M}_{1,k}(\mathbb{F}_p)$ has an orbit of size at least $\frac{\log(p)}{\log|\lambda|} + O_g(1)$, where λ is the eigenvalue of largest modulus of g when viewed as an element of $\mathrm{GL}_2(\mathbb{Z})$.

- M. de Courcy-Ireland and S. Lee [15] verify strong approximation for the Markoff surface for all primes $p < 3000$. Additionally, they completely characterize the orbit structure of the degenerate Cayley cubic, $\mathcal{M}_{1,4}(\mathbb{F}_p)$, providing both the number of orbits as well as their sizes, given in terms of divisors of p^2-1.
- M. de Courcy-Ireland and M. Magee [16] demonstrate that the eigenvalues of the family of Markoff graphs modulo p converge to the Kesten-McCay measure, which is a heuristic indicator that Markoff graphs are suitably "random". This also provides a (very) weak bound on the spectral gap of such graphs.
- M. de Courcy-Ireland [14] shows that if $p \equiv 1 \pmod 4$ or if $p \equiv 1, 2$ or $4 \pmod 7$, then the Markoff graph mod p is not planar.
- A. Gamburd , M. Magee and R. Ronan [18] prove that the counting function for the number of integer solutions on $x_1^2 + \cdots + x_n^2 = ax_1 \cdots x_n + k$, excluding potential exceptional sets, is asymptotic to a constant multiple of $(\log R)^\beta$.

3. Tri-Involutive K3 (TIK3) Surfaces

Definition 3.1. A *Tri-Involutive K3* (TIK3) *Surface* is a surface[2]

$$\mathcal{W} = \{\overline{F} = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

defined by a $(2,2,2)$-form

$$\overline{F}(X_1, X_2; Y_1, Y_2; Z_1, Z_2) \in K[X_1, X_2; Y_1, Y_2; Z_1, Z_2]. \tag{5}$$

For distinct $i, j \in \{1,2,3\}$, we denote the various projections of $\mathcal{W}$ onto one or two copies of $\mathbb{P}^1$ by

$$\pi_i : \mathcal{W} \longrightarrow \mathbb{P}^1 \quad \text{and} \quad \pi_{ij} : \mathcal{W} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

We say that the TIK3 is *non-degenerate* if it satisfies the following two conditions:

(i) The projection maps $\pi_{12}, \pi_{13}, \pi_{23}$ are finite.[3]

(ii) The generic fibers of the projection maps π_1, π_2, π_3 are smooth curves, in which case the smooth fibers are necessarily curves of genus 1, since they are $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

To ease notation, we write $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, and we let

$$F(x,y,z) = \overline{F}(x,1;y,1;z,1).$$

[2]We note that our terminology is something of a misnomer, since we do not insist that our surfaces be smooth.

[3]We note that $\pi_{12}, \pi_{13}, \pi_{23}$ are finite if and only if their fibers are 0-dimensional, in which case they are maps of degree 2.

Then $\mathcal{W}$ is the closure in $(\mathbb{P}^1)^3$ of the affine surface, which by abuse of notation we also denote by $\mathcal{W}$,

$$\mathcal{W} : F(x, y, z) = 0.$$

Definition 3.2. We denote the fibers of $\pi_1, \pi_2, \pi_3 : \mathcal{W} \to \mathbb{P}^1$ over points $x_0, y_0, z_0 \in \mathbb{P}^1$ by, respectively,

$$\mathcal{W}_{x_0}^{(1)} = \pi_1^{-1}(x_0), \qquad \mathcal{W}_{y_0}^{(2)} = \pi_2^{-1}(y_0), \qquad \mathcal{W}_{z_0}^{(3)} = \pi_3^{-1}(z_0).$$

For $P = (x_P, y_P, z_P) \in \mathcal{W}$, we let

$$\mathcal{W}_P^{(1)} = \mathcal{W}_{x_P}^{(1)}, \quad \mathcal{W}_P^{(2)} = \mathcal{W}_{y_P}^{(2)}, \quad \mathcal{W}_P^{(3)} = \mathcal{W}_{z_P}^{(3)}.$$

Definition 3.3. Let $\mathcal{W}$ be a non-degenerate TIK3. For distinct $i, j, k \in \{1, 2, 3\}$, we write

$$\sigma_k : \mathcal{W} \longrightarrow \mathcal{W} \tag{6}$$

for the involution that swaps the sheets of π_{ij}, i.e., $\sigma_k \in \operatorname{Aut}(\mathcal{W})$ is the unique non-identity automorphism satisfying

$$\pi_{ij} \circ \sigma_k = \pi_{ij}.$$

The automorphism group of a TIK3 surface $\mathcal{W}$ contains the non-commuting involutions $\sigma_1, \sigma_2, \sigma_3$, and depending on the symmetries of $\mathcal{W}$'s defining polynomial F, the automorphism group may contain additional automorphisms. Typical examples include symmetry in x, y, z that allows permutation of the coordinates, and power symmetry that allows the signs of two of x, y, z to be reversed. For example, the Markoff equation (1) permits these extra automorphisms; and in Section 7 we consider analogous TIK3 surfaces. In any case, we will be interested in subgroups of the automorphism group that move points around individual fibers.

Definition 3.4. Let $\mathcal{G} \subseteq \operatorname{Aut}(\mathcal{W})$ be a group of automorphisms of $\mathcal{W}$. We define the associated *fibral automorphism groups* by

$$\begin{aligned}
\mathcal{G}^{(1)} &= \bigl\{\varphi \in \mathcal{G} : \varphi(\mathcal{W}_x^{(1)}) = \mathcal{W}_x^{(1)} \text{ for all } x \in \mathbb{P}^1\bigr\},\\
\mathcal{G}^{(2)} &= \bigl\{\varphi \in \mathcal{G} : \varphi(\mathcal{W}_y^{(2)}) = \mathcal{W}_y^{(2)} \text{ for all } y \in \mathbb{P}^1\bigr\},\\
\mathcal{G}^{(3)} &= \bigl\{\varphi \in \mathcal{G} : \varphi(\mathcal{W}_z^{(3)}) = \mathcal{W}_z^{(3)} \text{ for all } z \in \mathbb{P}^1\bigr\}.
\end{aligned}$$

For example, if $\{i, j, k\} = \{1, 2, 3\}$, then $\sigma_i, \sigma_j \in \mathcal{G}^{(k)}$, since σ_i and σ_j map the k-fiber to itself.

Definition 3.5. Let $\mathcal{G} \subseteq \operatorname{Aut}(\mathcal{W})$ be a group of automorphisms of $\mathcal{W}$, and let $P_0 = (x_0, y_0, z_0) \in \mathcal{W}(K)$. The *$\mathcal{G}$-orbit of P* is

$$\mathcal{G} \cdot P = \bigl\{\varphi(P) : \varphi \in \mathcal{G}\bigr\}.$$

The *fibral $\mathcal{G}$-orbits of P* are

$$\mathcal{G}^{(k)} \cdot P = \bigl\{\varphi(P) : \varphi \in \mathcal{G}^{(k)}\bigr\} \quad \text{for } k = 1,2,3.$$

4. A strategy for proving that $\mathcal{W}(\mathbb{F}_q)$ has a large $\mathcal{G}$-connected component

In this section we consider a TIK3-surface $\mathcal{W}$ defined over a finite field $\mathbb{F}_q$, and a group of automorphisms $\mathcal{G} \subseteq \operatorname{Aut}(\mathcal{W})$.

Definition 4.1. Let $t \in \mathbb{P}^1(\mathbb{F}_q)$, and let $i \in \{1,2,3\}$. We say that the fiber $\mathcal{W}_t^{(i)}(\mathbb{F}_q)$ is *$\mathcal{G}$-fiber connected* if $\mathcal{G}^{(i)}$ acts transitively on $\mathcal{W}_t^{(i)}(\mathbb{F}_q)$. Following terminology from [7], we define the *$\mathcal{G}$-cage of $\mathcal{W}(\mathbb{F}_q)$* to be the set

$$\begin{aligned}&\mathsf{Cage}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)\\ &\qquad = \left\{ P \in \mathcal{W}(\mathbb{F}_q) : \begin{array}{l}\text{at least one of } \mathcal{W}_P^{(1)}(\mathbb{F}_q),\ \mathcal{W}_P^{(2)}(\mathbb{F}_q),\\ \text{and } \mathcal{W}_P^{(3)}(\mathbb{F}_q) \text{ is } \mathcal{G}\text{-fiber connected}\end{array}\right\}.\end{aligned}$$

We denote the set of $\mathcal{G}$-connected fibers by

$$\mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) = \left\{ \mathcal{W}_t^{(i)}(\mathbb{F}_q) : \begin{array}{l} i \in \{1,2,3\},\ t \in \mathbb{P}^1(\mathbb{F}_q),\\ \mathcal{W}_t^{(i)}(\mathbb{F}_q) \text{ is } \mathcal{G}\text{-fiber connected}\end{array}\right\}.$$

With this notation, an alternative description of the cage is as the union of the points in the fibers in $\mathsf{ConnFib}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$.

In [7], to prove that the Markoff graph $\mathcal{M}_{3,0}(\mathbb{F}_q)$ is connected, they first show that the associated cage is connected. This is done via a process that jumps from one connected fiber to another using a version of the following property:

Definition 4.2. We say that $\mathcal{W}(\mathbb{F}_q)$ has the *fiber-jumping property* if for all fibers $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{W}(\mathbb{F}_q)$ there exists a $\mathcal{G}$-connected fiber $\mathcal{F}_3 \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ satisfying

$$\mathcal{F}_1 \cap \mathcal{F}_3 \ne \emptyset \quad \text{and} \quad \mathcal{F}_2 \cap \mathcal{F}_3 \ne \emptyset.$$

As described in [7], the fiber-jumping property implies that the cage is connected. For the convenience of the reader, we recall the short proof.

Proposition 4.3. *Suppose that $\mathcal{W}(\mathbb{F}_q)$ has the fiber-jumping property. Then for all $P, Q \in \mathsf{Cage}_{\mathcal{G}}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ there exists an automorphism $\gamma \in \mathcal{G}$ such that $\gamma(Q) = P$.*

Proof. The fact that P and Q are in the $\mathcal{G}$-cage means that they lie on connected fibers, so we can find indices i and j so that

$$\mathcal{G}^{(i)} \cdot P = \mathcal{W}_P^{(i)}(\mathbb{F}_q) \quad \text{and} \quad \mathcal{G}^{(j)} \cdot Q = \mathcal{W}_Q^{(j)}(\mathbb{F}_q). \tag{7}$$

We apply the assumption that $\mathcal{W}(\mathbb{F}_q)$ has the fiber-jumping property to the fibers $\mathcal{W}_P^{(i)}(\mathbb{F}_q)$ and $\mathcal{W}_Q^{(j)}(\mathbb{F}_q)$. This allows us to find a connected fiber $\mathcal{F} \in \mathsf{ConnFib}\big(\mathcal{W}(\mathbb{F}_q)\big)$ satisfying

$$\mathcal{W}_P^{(i)}(\mathbb{F}_q) \cap \mathcal{F} \neq \emptyset \quad \text{and} \quad \mathcal{W}_Q^{(j)}(\mathbb{F}_q) \cap \mathcal{F} \neq \emptyset. \tag{8}$$

We choose any point $R \in \mathcal{F}$. The connectivity of $\mathcal{F}$ tells us that $\mathcal{F} = \mathcal{W}_R^{(k)}(\mathbb{F}_q) = \mathcal{G}^{(k)} \cdot R$ for some index k. Then (7) and (8) say that we can find points

$$S \in \mathcal{G}^{(i)} \cdot P \cap \mathcal{G}^{(k)} \cdot R \quad \text{and} \quad T \in \mathcal{G}^{(j)} \cdot Q \cap \mathcal{G}^{(k)} \cdot R.$$

In particular, there are automorphisms $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathcal{G}$ satisfying

$$S = \gamma_1 P = \gamma_2 R \quad \text{and} \quad T = \gamma_3 Q = \gamma_4 R.$$

This yields

$$P = \gamma_1^{-1}\gamma_2 R = \gamma_1^{-1}\gamma_2\gamma_4^{-1}\gamma_3 Q,$$

which completes the proof that $P \in \mathcal{G} \cdot Q$. □

The strategy that is employed in [7] to prove that the large component of the Markoff graph $\mathcal{M}_{3,0}(\mathbb{F}_q)$ is connected has several steps. We reformulate these steps for TIK3-surfaces, retaining (and expanding on) their chess terminology.

Setting the board (Cage connectivity):
The cage $\mathsf{Cage}_{\mathcal{G}}\big(\mathcal{W}(\mathbb{F}_q)\big)$ is $\mathcal{G}$-connected.

End game (Large fibral orbits):
Let $P \in \mathcal{W}_t^{(i)}(\mathbb{F}_q)$ be a point whose fibral orbit $\mathcal{G}^{(i)} \cdot P$ is moderately large. Then $\mathcal{G}^{(i)} \cdot P$ contains a point of the cage, i.e., it intersects a $\mathcal{G}$-connected fiber.

Middle game (Small fibral orbits):
Let $P \in \mathcal{W}_t^{(i)}(\mathbb{F}_q)$ be a point whose fibral orbit $\mathcal{G}^{(i)} \cdot P$ is of small, but non-negligible, size. Then $\mathcal{G}^{(i)} \cdot P$ contains a point lying in a fibral orbit of strictly larger size.

Opening (Tiny fibral orbits):
There are no non-trivial points $P \in \mathcal{W}_t^{(i)}(\mathbb{F}_q)$ whose fibral orbit $\mathcal{G}^{(i)} \cdot P$ is tiny.

Remark 4.4 (The Bourgain–Gamburd–Sarnak Connectivity Proof for the Markoff Equation)**.** We briefly sketch the connectivity proof for

$$\mathcal{M}^*(\mathbb{F}_p) = \mathcal{M}_{3,0}(\mathbb{F}_p) \smallsetminus (0,0,0)$$

in [7]. They prove connectivity using the subgroup $\mathcal{G} \subset \operatorname{Aut}(\mathcal{M}_{3,0})$ generated by the compositions

$$\rho^{(i)} = \varphi_i \circ \tau_{jk}, \quad \text{where } \{i,j,k\} = \{1,2,3\}.$$

They call $\rho^{(i)}$ a *rotation*, since it acts on the fibers $(\mathcal{M}_{3,0})_t^{(i)}$ via a 2-by-2 (rotation) matrix acting on the jk-coordinates. Writing $\rho_t^{(i)}$ for the restriction of $\rho^{(i)}$ to this fiber, they note that the order of $\rho_t^{(i)}$ divides one of $p-1$, p, or $p+1$, with the exact order depending on the eigenvalues of the matrix $\rho_t^{(i)}$. It follows that

$$(\mathcal{M}_{3,0})_t^{(i)}(\mathbb{F}_p) \subset \mathsf{Cage}\big(\mathcal{M}_{3,0}(\mathbb{F}_p)\big) \quad \Longleftrightarrow \quad \rho_t^{(i)} \text{ has maximal order.}$$

The first step in proving that $\mathcal{M}^*(\mathbb{F}_p)$ is $\mathcal{G}$-connected is an argument that uses curve coverings, point counting, and inclusion/exclusion to show that $\mathcal{M}_{3,0}(\mathbb{F}_p)$ has the fiber jumping property for $\mathcal{G}$. It follows that $\mathsf{Cage}_{\mathcal{G}}\big(\mathcal{M}_{3,0}(\mathbb{F}_p)\big)$ is connected, cf. Proposition 4.3. They then use a similar argument for the endgame, where a fiber is deemed large if it has $p^{1/2+\epsilon}$ points.

Next they consider the middle game, which consists of points whose (small) fibral orbit has at least p^ϵ points. This comes down to showing that certain equations have few solutions whose coordinates are elements of $\mathbb{F}_p^*$ of small order. They provide three proofs of the required statement, one via Stepanov's auxiliary polynomial proof of Weil's conjecture for curves over $\mathbb{F}_p$, one using directly a sharp estimate due to Corvaja and Zannier [13] for the gcd of polynomials over finite fields, and one using a projective Szemeredi-Trotter theorem due to Bourgain [6]. Indeed, they can handle the middle game for even smaller fibral components provided that p^2-1 does not have too many prime divisors.

Finally, for the opening, they observe that finite orbits in $\mathcal{M}_{a,k}(\overline{\mathbb{Q}})$ will create small orbits in $\mathcal{M}_{a,k}(\mathbb{F}_p)$ for infinitely many p. However, in their case $\mathcal{M}_{3,0}(\overline{\mathbb{Q}})$ contains no finite orbits other than $\big\{(0,0,0)\big\}$, so this is not a problem. They next show that every point $P \in \mathcal{M}^*(\mathbb{F}_p)$ lies in a fibral component containing at least $(\log_{20} p)^{1/3}$ points. This and some further calculations suffice to prove that $\mathcal{M}^*(\mathbb{F}_p)$ is $\mathcal{G}$-connected unless p^2-1 is very smooth, i.e., is a product of a large number of small primes. (Conjecturally, there are only finitely many such primes.)

Remark 4.5 (Fiber Jumping and Cage Connectivity for TIK3-Surfaces)**.** As explained in Remark 4.4, Bourgain, Gamburd, and Sarnak [7] prove that the Markoff equation $\mathcal{M}_{3,0}(\mathbb{F}_p) \smallsetminus \big\{(0,0,0\big\}$ is $\mathcal{G}$-connected by first verifying the fiber-jumping property, which sets the board by implying that the cage is $\mathcal{G}$-connected. Later we will give an example showing that the analogous statement need not be true for TIK3 surfaces. More precisely, in Example 12.1 we describe a TIK3-surface $\mathcal{W}$ such that $\mathcal{W}(\mathbb{F}_{53})$ has one large $\mathcal{G}$-connected component $\mathcal{W}^*(\mathbb{F}_{53})$ containing 3456 points, but $\mathcal{W}^*(\mathbb{F}_{53})$ does not have the $\mathcal{G}$-fiber-jumping property. More precisely, the $\mathcal{G}$-connected fibers in $\mathcal{W}(\mathbb{F}_{53})$ form two connected components, so any proof that $\mathcal{W}^*(\mathbb{F}_{53})$ is $\mathcal{G}$-connected must find a way to connect points in $\mathsf{ConnFib}\big(\mathcal{W}(\mathbb{F}_{53})\big)$ that uses points that do not lie on a $\mathcal{G}$-connected fiber, i.e., using points that are not in the cage. Of course, the prime $p = 53$ is not huge, so our example may simply be a small number phenomenon. However, other examples suggest that the number of fibral components in a TIK3 cage tends to be smaller than the number of fibral components in a Markoff surface cage. So a proof that TIK3 surfaces over finite fields have large $\mathcal{G}$-connected components may need to find a way to expand the cage in order to fit it into a $\mathcal{G}$-connected set that can be used for the "setting the board" step.

In addition, the issue concerning smoothness of fibral group orders that arises in the method of BGS will be exacerbated for TIK3 surfaces. The analogous rotations (translations) on a TIK3 surface come from the actions of elliptic curves on homogeneous spaces. These actions are translations by a point whose order can range from $p + 1 - 2\sqrt{p}$ to $p + 1 + 2\sqrt{p}$. So now we are not concerned with smoothness of only $p \pm 1$, but instead with the smoothness of all numbers within this range. Ideally, we would like to restrict to values of p for which this range of numbers contains no smooth numbers, but there are unlikely to be infinitely many such p.

5. A brief survey of related work on tri-involutive K3 surfaces

We briefly describe some earlier work on the geometry and arithmetic of TIK3 surfaces. Wang [25] explicitly constructed canonical heights on TIK3 surfaces defined over number fields associated to the infinite order automorphisms $\sigma_i \circ \sigma_j$, similar to those constructed in [23] for K3 surfaces having two involutions. Baragar [2, 3, 4] further studied these height functions and asked, in particular, whether they fit together to form a vector canonical height. Kawaguchi [19] answered this in the negative for certain K3 surfaces, and Cantat and Dujardin [10] completely characterized the surfaces on which vector canonical heights exist.

We next state a recent result regarding finite orbits on TIK3 surfaces in charateristic 0.

Theorem 5.1 ([10, Cantat–Dujardin])**.** *Let $\mathcal{W}/\mathbb{C}$ be a TIK3 surface, and let $\langle\sigma_1,\sigma_2,\sigma_3\rangle \subseteq \mathrm{Aut}(\mathcal{W})$ be the subgroup of $\mathcal{W}$ generated by the three involutions $\sigma_1,\sigma_2,\sigma_3$. Then*

$$\bigl\{P \in \mathcal{W}(\mathbb{C}) : \textit{the } \langle\sigma_1,\sigma_2,\sigma_3\rangle\textit{-orbit of } P \textit{ is finite}\bigr\}$$

is a finite set.

Proof. This is a special case of the results in [10], since in the language of [10], the TIK3-surface $\mathcal{W}$ and its group of automorphisms $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ do not form a Kummer group, and $\mathcal{W}$ contains no $\langle\sigma_1,\sigma_2,\sigma_3\rangle$-invariant curves. □

Finally, we mention Cantat's fundamental paper [9], although it is not specifically about TIK3 surfaces. Let $\varphi : \mathcal{X} \to \mathcal{X}$ be an automorphism of positive entropy of a K3 surface defined over $\mathbb{C}$, e.g., $\sigma_i \circ \sigma_j$ for a TIK3 surface. Then Cantat proves that there exists a unique invariant probability measure μ with maximal entropy, that (φ,μ) is measurably conjugate to a Bernoulli shift, and that μ gives the asymptotic distribution of periodic points.

6. The incidence graph of the fibers of a TIK3 surface

Definition 6.1. A TIK3 surface has three fibral directions associated to the three projections onto $\mathbb{P}^1$. For expositional convenience, we will say that fibers corresponding to different projections are (*pairwise*) *orthogonal* to one another, while fibers corresponding to the same projection are *parallel*. So for example, the fibers $\mathcal{W}^{(1)}_{x_0}$ and $\mathcal{W}^{(2)}_{y_0}$ are orthogonal, while the fibers $\mathcal{W}^{(1)}_{x_0}$ and $\mathcal{W}^{(1)}_{x_1}$ are parallel.

Remark 6.2. Distinct parallel fibers clearly do not intersect, while orthogonal fibers in $\mathcal{W}(\mathbb{F}_q)$ may intersect in 0, 1, or 2 points. For example, if $x_0, y_0 \in \mathbb{P}^1(\mathbb{F}_q)$, then

$$\Bigl(\mathcal{W}^{(1)}_{x_0}(\mathbb{F}_q) \cap \mathcal{W}^{(2)}_{y_0}(\mathbb{F}_q)\Bigr) = \bigl\{(x_0,y_0,z) : F(x_0,y_0,z)=0\bigr\}.$$

Thus the intersection is non-empty if and only if a certain quadratic form has a solution in $\mathbb{P}^1(\mathbb{F}_q)$.

Our goal in this section is to give an easily verifiable condition which ensures that, given two orthogonal fibers $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathcal{W}(\mathbb{F}_q)$, there is a third fiber $\mathcal{F}_3 \subset \mathcal{W}(\mathbb{F}_q)$ satisfying

$$\mathcal{F}_1 \cap \mathcal{F}_3 \ne \emptyset \quad\text{and}\quad \mathcal{F}_2 \cap \mathcal{F}_3 \ne \emptyset.$$

In more evocative terms, although the union $\mathcal{F}_1 \cup \mathcal{F}_2$ of two orthogonal fibers may be "disconnected," there is a third fiber so that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a "connected" set of orthogonal fibers. See Figure 1.

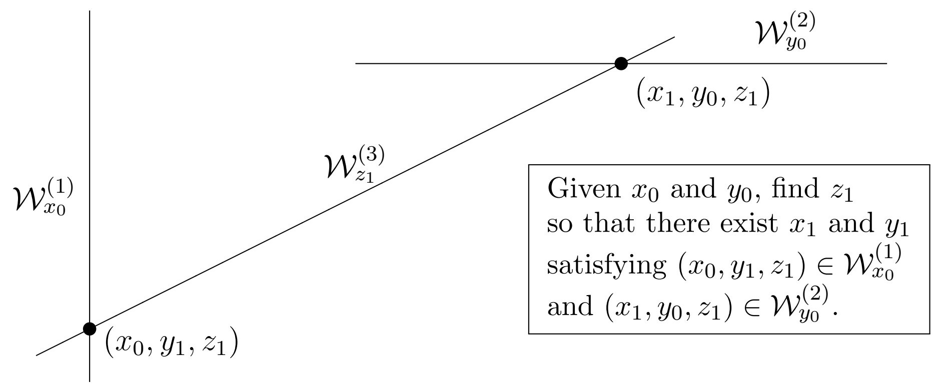

FIGURE 1. Finding a fiber $\mathcal{W}^{(3)}_{z_1}$ that intersects two given fibers $\mathcal{W}^{(1)}_{x_0}$ and $\mathcal{W}^{(2)}_{y_0}$

Definition 6.3. For $x_0, y_0, z_0 \in \mathbb{P}^1$, we define linking sets that describe how to link two given fibers via a third fiber.

$$\begin{aligned}
\mathcal{L}^{(1)}_{y_0,z_0} &= \bigl\{x \in \mathbb{P}^1 : \mathcal{W}^{(2)}_{y_0} \cap \mathcal{W}^{(1)}_{x} \ne \emptyset \text{ and } \mathcal{W}^{(3)}_{z_0} \cap \mathcal{W}^{(1)}_{x} \ne \emptyset\bigr\},\\
\mathcal{L}^{(2)}_{x_0,z_0} &= \bigl\{y \in \mathbb{P}^1 : \mathcal{W}^{(1)}_{x_0} \cap \mathcal{W}^{(2)}_{y} \ne \emptyset \text{ and } \mathcal{W}^{(3)}_{z_0} \cap \mathcal{W}^{(2)}_{y} \ne \emptyset\bigr\},\\
\mathcal{L}^{(3)}_{x_0,y_0} &= \bigl\{z \in \mathbb{P}^1 : \mathcal{W}^{(1)}_{x_0} \cap \mathcal{W}^{(3)}_{z} \ne \emptyset \text{ and } \mathcal{W}^{(2)}_{y_0} \cap \mathcal{W}^{(3)}_{z} \ne \emptyset\bigr\}.
\end{aligned}$$

Thus for example, the points in $\mathcal{L}^{(3)}_{x_0,y_0}$ tell us which z fibers can be used to link the $x = x_0$ fiber with the $y = y_0$ fiber.

Definition 6.4. For $x_0, y_0, z_0 \in \mathbb{P}^1$, we define the following curves that are useful in creating fibral links:

$$\begin{aligned}
\mathcal{C}^{(1)}_{y_0,z_0} &= \bigl\{(x,y,z) \in (\mathbb{P}^1)^3 : F(x,y_0,z) = F(x,y,z_0) = 0\bigr\},\\
\mathcal{C}^{(2)}_{x_0,z_0} &= \bigl\{(x,y,z) \in (\mathbb{P}^1)^3 : F(x_0,y,z) = F(x,y,z_0) = 0\bigr\},\\
\mathcal{C}^{(3)}_{x_0,y_0} &= \bigl\{(x,y,z) \in (\mathbb{P}^1)^3 : F(x_0,y,z) = F(x,y_0,z) = 0\bigr\}.
\end{aligned}$$

We note that the curve $\mathcal{C}^{(1)}_{y_0,z_0}$ is the intersection in $(\mathbb{P}^1)^3$ of a hypersurface of type $(2,0,2)$ and a hypersurface of type $(2,2,0)$, and similarly for $\mathcal{C}^{(2)}_{x_0,z_0}$ and $\mathcal{C}^{(3)}_{x_0,y_0}$. (See Lemma 6.6 for an estimate of the genera of these curves.)

Theorem 6.5 (K3 Analogue of [8, Proposition 6])**.** *Let K be a field, and let $x_0, y_0, z_0 \in \mathbb{P}^1(K)$.*

(a) *There are* surjective *maps*

$$\mathcal{C}^{(1)}_{y_0,z_0}(K) \xrightarrow{(x,y,z)\mapsto x} \mathcal{L}^{(1)}_{y_0,z_0}(K),$$

$$\mathcal{C}^{(2)}_{x_0,z_0}(K) \xrightarrow{(x,y,z)\mapsto y} \mathcal{L}^{(2)}_{x_0,z_0}(K),$$
$$\mathcal{C}^{(3)}_{x_0,y_0}(K) \xrightarrow{(x,y,z)\mapsto z} \mathcal{L}^{(3)}_{x_0,y_0}(K).$$

(b) *Assume that $q \ge 100$. Then*

$$\mathcal{L}^{(1)}_{y_0,z_0}(\mathbb{F}_q) \ne \emptyset, \qquad \mathcal{L}^{(2)}_{x_0,z_0}(\mathbb{F}_q) \ne \emptyset, \qquad \mathcal{L}^{(3)}_{x_0,y_0}(\mathbb{F}_q) \ne \emptyset.$$

Proof. (a) By symmetry, it suffices to prove that the first map is well-defined and surjective. Let $(x,y,z) \in \mathcal{C}^{(1)}_{y_0,z_0}(K)$. By definition of $\mathcal{C}^{(1)}_{y_0,z_0}$, this means that

$$F(x,y_0,z) = F(x,y,z_0) = 0, \quad \text{and thus} \quad (x,y_0,z),\ (x,y,z_0) \in \mathcal{W}(K).$$

Hence

$$(x,y_0,z) \in \mathcal{W}^{(2)}_{y_0}(K) \cap \mathcal{W}^{(1)}_{x}(K) \quad \text{and} \quad (x,y,z_0) \in \mathcal{W}^{(3)}_{z_0}(K) \cap \mathcal{W}^{(1)}_{x}(K),$$

which by definition of $\mathcal{L}^{(1)}_{y_0,z_0}$ shows that $x \in \mathcal{L}^{(1)}_{y_0,z_0}(K)$. This completes the proof that the projection map

$$\pi_1 : \mathcal{C}^{(1)}_{y_0,z_0}(K) \longrightarrow \mathcal{L}^{(1)}_{y_0,z_0}(K) \tag{9}$$

is well-defined.

To prove surjectivity, we start with some $x \in \mathcal{L}^{(1)}_{y_0,z_0}(K)$. By definition of $\mathcal{L}^{(1)}_{y_0,z_0}$, this means that we can find points

$$(x,y_0,z_1) \in \mathcal{W}^{(2)}_{y_0}(K) \cap \mathcal{W}^{(1)}_{x}(K) \quad \text{and} \quad (x,y_1,z_0) \in \mathcal{W}^{(3)}_{z_0}(K) \cap \mathcal{W}^{(1)}_{x}(K).$$

Then the definition of $\mathcal{C}^{(1)}_{y_0,z_0}$ tells us that

$$(x,y_1,z_1) \in \mathcal{C}^{(1)}_{y_0,z_0}(K).$$

We have thus constructed a point in $\mathcal{C}^{(1)}_{y_0,z_0}(K)$ whose image in $\mathcal{L}^{(1)}_{y_0,z_0}(K)$ is x, which completes the proof that the projection map (9) is surjective.
(b) We use (a) with $K = \mathbb{F}_q$. Again by symmetry, it suffices to prove the first assertion. And from the surjectivity of the map in (a), it suffices to prove that $\mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q)$ is not empty.

We let $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}$ be a non-singular model for $\mathcal{C}^{(1)}_{y_0,z_0}$ (or more generally for any one of its irreducible components if it happens to be reducible), so in particular we have a surjection

$$\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q) \longrightarrow \mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q).$$

Then the Weil estimate gives the inequality

$$\#\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q) \ge q + 1 - 2 \cdot \bigl(\operatorname{genus} \widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr) \cdot \sqrt{q}. \tag{10}$$

In particular, we see that

$$q+1 > 2\cdot\left(\operatorname{genus}\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\right)\cdot\sqrt{q} \quad\Longrightarrow\quad \widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}(\mathbb{F}_q)\neq\emptyset. \qquad (11)$$

Lemma 6.6, whose proof we defer for the moment, says that the genus of $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}$ is at most 5. Hence (10) and (11) imply that $\mathcal{C}^{(1)}_{y_0,z_0}(\mathbb{F}_q)$ is non-empty provided $q+1>10\sqrt{q}$, which is true for all $q>100$. $\square$

We now prove the genus estimate used in the proof of Theorem 6.5.

Lemma 6.6. *Let $\mathcal{W}$ be a non-degenerate TIK3 surface. Then the irreducible components of each of the curves in Definition 6.4 has geometric genus at most* 5.

Proof. We work over an algebraically closed field. By symmetry, it suffices to fix $y_0,z_0\in\mathbb{P}^1$ and to consider the curve $\mathcal{C}^{(1)}_{y_0,z_0}$. We let F be the $(2,2,2)$-form that defines the non-degenerate TIK3 surface $\mathcal{W}$. We define a projection map

$$\pi:\mathcal{C}^{(1)}_{y_0,z_0}\longrightarrow\mathbb{P}^1,\quad \pi(x,y,z)=x.$$

Keeping in mind that y_0 and z_0 are fixed, for $x_1\in\mathbb{P}^1$ we have

$$\pi^{-1}(x_1)=\bigl\{(y,z)\in(\mathbb{P}^1)^2 : F(x_1,y_0,z)=F(x_1,y,z_0)=0\bigr\}.$$

The equations for y and z are independent, so we find that

$$\#\pi^{-1}(x_1)=\#\bigl\{z\in\mathbb{P}^1 : F(x_1,y_0,z)=0\bigr\}\cdot\#\bigl\{y\in\mathbb{P}^1 : F(x_1,y,z_0)=0\bigr\}.$$

The non-degeneracy assumption tells us that $F(x_1,y_0,z)$ and $F(x_1,y,z_0)$ are not identically 0, so they are non-trivial quadratic forms in, respectively, z and y. As such, they have either 1 or 2 roots, and we can determine which is the case by computing an appropriate discriminant:

$$\#\bigl\{z\in\mathbb{P}^1 : F(x_1,y_0,z)=0\bigr\}=\begin{cases}1 & \text{if } \operatorname{Disc}_z F(x_1,y_0,z)=0,\\ 2 & \text{if } \operatorname{Disc}_z F(x_1,y_0,z)\neq 0.\end{cases}$$

$$\#\bigl\{y\in\mathbb{P}^1 : F(x_1,y,z_0)=0\bigr\}=\begin{cases}1 & \text{if } \operatorname{Disc}_y F(x_1,y,z_0)=0,\\ 2 & \text{if } \operatorname{Disc}_y F(x_1,y,z_0)\neq 0.\end{cases}$$

Combining these estimates yields the following formulas

$\#\pi^{-1}(x_1)$	$\operatorname{Disc}_y F(x_1,y,z_0)$	$\operatorname{Disc}_z F(x_1,y_0,z)$
4	$\neq 0$	$\neq 0$
2	$=0$	$\neq 0$
2	$\neq 0$	$=0$
1	$=0$	$=0$

We next observe that $\operatorname{Disc}_y F(x,y,z_0)$ is a degree 4 form in x, and thus has at most 4 roots in $\mathbb{P}^1$ when considered as a polynomial in x; and similarly for $\operatorname{Disc}_z F(x,y_0,z)$. So there are at most 8 points $x_1\in\mathbb{P}^1$ with $\#\pi^{-1}(x_1)=2$. Further, each time we get an x_1 with $\#\pi^{-1}(x_1)=1$, we see that 2 of those 8 potential values of x_1 coalesce into 1 value. So if we let

$$\begin{aligned} A&=\#\bigl\{x_1\in\mathbb{P}^1:\pi^{-1}(x_1)=2\bigr\},\\ B&=\#\bigl\{x_1\in\mathbb{P}^1:\pi^{-1}(x_1)=1\bigr\}, \end{aligned} \tag{12}$$

then we see that

B	0	1	2	3	4
A	≤ 8	≤ 6	≤ 4	≤ 2	$= 0$

(13)

We assume for the moment that $\mathcal{C}^{(1)}_{y_0,z_0}$ is irreducible,[4] and we let

$$\lambda:\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\longrightarrow \mathcal{C}^{(1)}_{y_0,z_0}$$

be a desingularization of $\mathcal{C}^{(1)}_{y_0,z_0}$, so the geometric genus of $\mathcal{C}^{(1)}_{y_0,z_0}$ is simply the genus of $\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}$. We use the Riemann–Hurwitz genus formula

$$2\operatorname{genus}\bigl(\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr)-2=-2\deg(\pi\circ\lambda)+\sum_{x_1\in\mathbb{P}^1}\Bigl(\deg(\pi\circ\lambda)-\#(\pi\circ\lambda)^{-1}(x_1)\Bigr).$$

Substituting

$$\deg\pi\circ\lambda=\deg(\pi)\cdot\deg(\lambda)=4\cdot 1=4,$$

we get

$$\begin{aligned} \operatorname{genus}\bigl(\widetilde{\mathcal{C}^{(1)}_{y_0,z_0}}\bigr) &= -3+\frac12\sum_{\substack{x_1\in\mathbb{P}^1\\ \#(\pi\circ\lambda)^{-1}(x_1)<4}}\Bigl(4-\#(\pi\circ\lambda)^{-1}(x_1)\Bigr)\\ &\le -3+\frac12\sum_{\substack{x_1\in\mathbb{P}^1\\ \#\pi^{-1}(x_1)<4}}\Bigl(4-\#\pi^{-1}(x_1)\Bigr)\\ &= -3+\#\bigl\{x_1\in\mathbb{P}^1:\#\pi^{-1}(x_1)=2\bigr\}\\ &\qquad +\frac32\#\bigl\{x_1\in\mathbb{P}^1:\#\pi^{-1}(x_1)=1\bigr\}\\ &= -3+A+\frac32 B\quad\text{using the notation in (12)},\\ &\le 5\quad\text{from (13), since the max is at }(A,B)=(8,0). \end{aligned}$$

[4]See Remark 6.7 for examples where $\mathcal{C}^{(1)}_{y_0,z_0}$ is reducible.

Finally, we note that if $\mathcal{C}^{(1)}_{y_0,z_0}$ is reducible, then the above argument works similarly, if we replace $\mathcal{C}^{(1)}_{y_0,z_0}$ with any of its irreducible components and note that now the map π has degree 1 or 2. This completes the proof of Lemma 6.6. $\square$

Remark 6.7. Let $\mathcal{W}$ be a TIK3 surface whose equation F is symmetric in y and z, i.e., $F(x,y,z) = F(x,z,y)$. Then for any $\xi \in K$ there is a factorization

$$F(x,\xi,z) - F(x,y,\xi) = F(x,z,\xi) - F(x,y,\xi) = (z-y)L(x,y,z),$$

where $L(x,y,z)$ has degree 1 in y and z. It follows the curve $\mathcal{C}^{(1)}_{\xi,\xi}$ described in Definition 6.4 is reducible, and indeed it is the union of two genus 1 curves, each of which is isomorphic to the fibral curve

$$\mathcal{W}^{(3)}_{\xi} \cong \bigl\{(x,y) \in \mathbb{A}^2 : F(x,y,\xi) = 0\bigr\}$$

7. Tri-Involutive Markoff-Type K3 (MK3) Surfaces

The Markoff equation (1) and many of its variants admit not only the involutions coming from the projections $\mathcal{M} \to \mathbb{A}^2$, they also admit sign-change involutions and coordinate permutations coming from the symmetry of the Markoff equation. We give a name to the TIK3 surfaces that have these extra automorphisms.

Definition 7.1. We let $\mathfrak{S}_3$, the symmetric group on 3 letters, act on $(\mathbb{P}^1)^3$ by permuting the coordinates, and we let the group

$$(\boldsymbol{\mu}_2^3)_1 := \bigl\{(\alpha,\beta,\gamma) : \alpha,\beta,\gamma \in \boldsymbol{\mu}_2 \text{ and } \alpha\beta\gamma = 1\bigr\} \tag{14}$$

act on $(\mathbb{P}^1)^3$ via sign changes,

$$\epsilon_{\alpha,\beta,\gamma}(x,y,z) = (\alpha x, \beta y, \gamma z). \tag{15}$$

In this way we obtain an embedding[5]

$$\mathcal{G}^\circ := (\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3 \hookrightarrow \operatorname{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1).$$

Definition 7.2. A *Markoff-type K3* (MK3) *surface* $\mathcal{W}$ is a TIK3 surface whose $(2,2,2)$-form (5) is invariant under the action of $\mathcal{G}^\circ$, i.e., the $(2,2,2)$-form F describing $\mathcal{W}$ satisfies

$$F(x,y,z) = F(-x,-y,z) = F(-x,y,-z) = F(x,-y,-z),$$
$$F(x,y,z) = F(z,x,y) = F(y,z,x) = F(x,z,y) = F(y,x,z) = F(z,y,x).$$

[5] We remark that $(\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3$ is isomorphic to $\mathfrak{S}_4$, but for our applications the group $\mathcal{G}^\circ$ appears more naturally as the semi-direct product.

Definition 7.3. Let $\mathcal{W}$ be an MK3 surface. We let

$$\begin{aligned}\mathcal{G}^\sigma &= \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset \operatorname{Aut}(\mathcal{W}),\\ \mathcal{G} &= \langle \text{group generated by } \mathcal{G}^\sigma \text{ and } \mathcal{G}^\circ \rangle \subset \operatorname{Aut}(\mathcal{W}).\end{aligned}$$

We suspect that the full automorphism group of a generic MK3-surface is $\mathcal{G}$; but as we shall see in Remark 9.6, some MK3-surfaces admit additional automorphisms. We start by describing some elementary properties of the group $\mathcal{G}$.

Proposition 7.4. *Let $\mathcal{W}$ be an MK3-surface, and let $\mathcal{G}^\circ$, $\mathcal{G}^\sigma$, and $\mathcal{G}$ be the subgroups of* $\operatorname{Aut}(\mathcal{W})$ *described in Definitions 7.1 and 7.3.*
(a) *$\mathcal{G}^\sigma$ is a normal subgroup of $\mathcal{G}$.*
(b) *$\mathcal{G} = \mathcal{G}^\circ \mathcal{G}^\sigma$.*

Proof. (a) Since $\mathcal{G}$ is defined to be the group generated by $\mathcal{G}^\circ$ and $\mathcal{G}^\sigma$, it suffices to show that $\mathcal{G}^\circ$ is contained in the normalizer of $\mathcal{G}^\sigma$. We let $\{i,j,k\} = \{1,2,3\}$, and for the purposes of this proof, we define transpositions and sign changes

$$\begin{aligned}\tau_{ij} &= \text{swap the } i \text{ and } j \text{ coordinates},\\ \epsilon_{ij} &= \text{multiply the } i \text{ and } j \text{ coordinates by } -1.\end{aligned}$$

Since $\mathfrak{S}_3$ is generated by transpositions and $(\boldsymbol{\mu}_2^3)_1$ is generated by the sign changes, it suffices to check that $\mathcal{G}^\sigma$ is normalized by the τ_{ij} and the ϵ_{ij}. This can be checked by an explicit computation, or alternatively we can use the defining property $\pi_{ij} \circ \sigma_k = \pi_{ij}$ of σ_k, where π_{ij} is the projection map; see Definition 3.3. Thus momentarily letting $\tau : (\mathbb{P}^1)^2 \to (\mathbb{P}^1)^2$ be the map that swaps the coordinates and $\epsilon_i : (\mathbb{P}^1)^2 \to (\mathbb{P}^1)^2$ be the map that changes the sign of the ith coordinate, we compute

$$\begin{aligned}\pi_{ij} \circ (\tau_{ij}^{-1} \circ \sigma_k \circ \tau_{ij}) &= \tau \circ \pi_{ij} \circ \sigma_k \circ \tau_{ij} = \tau \circ \pi_{ij} \circ \tau_{ij} = \pi_{ij},\\ \pi_{jk} \circ (\tau_{ik}^{-1} \circ \sigma_k \circ \tau_{ik}) &= \tau \circ \pi_{ij} \circ \sigma_k \circ \tau_{ik} = \tau \circ \pi_{ij} \circ \tau_{ik} = \pi_{jk}\\ \pi_{ij} \circ (\epsilon_{ij}^{-1} \circ \sigma_k \circ \epsilon_{ij}) &= \epsilon_{ij} \circ \pi_{ij} \circ \sigma_k \circ \epsilon_{ij} = \epsilon_{ij} \circ \pi_{ij} \circ \epsilon_{ij} = \epsilon_{ij}^2 \circ \pi_{ij} = \pi_{ij},\\ \pi_{ij} \circ (\epsilon_{ik}^{-1} \circ \sigma_k \circ \epsilon_{ik}) &= \epsilon_i \circ \pi_{ij} \circ \sigma_k \circ \epsilon_{ik} = \epsilon_i \circ \pi_{ij} \circ \epsilon_{ik} = \epsilon_i^2 \circ \pi_{ij} = \pi_{ij}.\end{aligned}$$

It follows from the definitions of the σ_i that

$$\begin{aligned}\tau_{ij}^{-1} \circ \sigma_k \circ \tau_{ij} &= \sigma_k, & \epsilon_{ij}^{-1} \circ \sigma_k \circ \epsilon_{ij} &= \sigma_k,\\ \tau_{ik}^{-1} \circ \sigma_k \circ \tau_{ik} &= \sigma_i, & \epsilon_{ik}^{-1} \circ \sigma_k \circ \epsilon_{ik} &= \sigma_k.\end{aligned}$$

Hence $\mathcal{G}^\circ$ normalizes $\mathcal{G}^\sigma$, and indeed, $(\boldsymbol{\mu}_2^3)_1$ is in the centralizer of $\mathcal{G}^\sigma$.
(b) By definition the group $\mathcal{G}$ is generated by $\mathcal{G}^\circ$ and $\mathcal{G}^\sigma$, and from (a), we know that $\mathcal{G}^\sigma$ is a normal subgroup of $\mathcal{G}$. It follows that every

element of $\mathcal{G}$ can be written as $\gamma\sigma$ with $\gamma \in \mathcal{G}^\circ$ and $\sigma \in \mathcal{G}^\sigma$. Hence $\mathcal{G} = \mathcal{G}^\circ\mathcal{G}^\sigma$. ☐

Proposition 7.5. *Let $\mathcal{W}/K$ be a (possibly degenerate) MK3-surface.*

(a) *There exist $a, b, c, d, e \in K$ so that the $(2,2,2)$-form F that defines $\mathcal{W}$ has the form*

$$\begin{aligned} F_{a,b,c,d,e}(x,y,z) = ax^2y^2z^2 + b(x^2y^2 + x^2z^2 + y^2z^2) \\ + cxyz + d(x^2+y^2+z^2) + e = 0. \end{aligned} \tag{16}$$

(b) *Let F be as in* (a). *Then $\mathcal{W}$ is a non-degenerate, i.e., the projections $\pi_{ij} : \mathcal{W} \to (\mathbb{P}^1)^2$ are quasi-finite, if and only if*

$$be \neq d^2 \quad \textit{and} \quad ad \neq b^2.$$

Remark 7.6. We can recover the classical (translated) Markoff equation for the surface $\mathcal{M}_{a,k}$ in Definition 1 as a special case of an $F_{a,b,c,d,e}$. Thus $\mathcal{M}_{a,k}$ is given by the affine equation

$$F_{0,0,-a,1,-k}(x,y,z) = x^2 + y^2 + z^2 - axyz - k = 0.$$

We note, however, that the Markoff equation is degenerate, despite the involutions being well-defined on the affine Markoff surface $\mathcal{M}_{a,k}$. This occurs because the involutions are not well-defined at some of the points at infinity in the closure of $\mathcal{M}_{a,k}$ in $(\mathbb{P}^1)^3$.

Proof of 7.5. (a) The space of $\mathfrak{S}_3$-invariant quadratic polynomials in $\mathbb{Z}[x,y,z]$ is spanned by the following 10 polynomials:

(1)	$x^2y^2z^2$	(2)	$xyz^2 + xy^2z + x^2yz$
(3)	xyz	(4)	$x^2y^2z + x^2yz^2 + xy^2z^2$
(5)	$x^2+y^2+z^2$	(6)	$x^2y^2 + x^2z^2 + y^2z^2$
(7)	$x^2y + x^2z + xy^2 + xz^2 + yz^2 + y^2z$		
(8)	$xy + xz + yz$	(9)	$x + y + z$
(10)	1		

Of these, the polynomials that are also invariant for the double-sign changes in $(\boldsymbol{\mu}_2^3)_1$ are (1), (3), (5), (6), and (10). Hence all $\big((\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3\big)$-invariant $(2,2,2)$-polynomials have the form indicated in (a).
(b) By symmetry, it suffices to consider π_{12} and σ_3. The map π_{12} is quasi-finite if and only if the fibers of the map π_{12} are 0-dimensional. Let $\overline{F}$ be the homogenization of the polynomial in (a). Then π_{12} is quasi-finite over the point

$$\big([\alpha,\beta],[\gamma,\delta]\big) \in \mathbb{P}^1 \times \mathbb{P}^1$$

if and only if the polynomial $F(\alpha,\beta;\gamma,\delta;X_3,Y_3)$ is not identically 0. Since

$$\Bigl(\text{the } X_3Y_3 \text{ term of } F(\alpha,\beta;\gamma,\delta;X_3,Y_3)\Bigr) = \alpha\beta\gamma\delta X_3Y_3,$$

we see that π_{12} is quasi-finite unless $\alpha\beta\gamma\delta = 0$. By the symmetry of F, it suffices to consider the cases that $\alpha = 0$ and $\beta = 0$.

If $\alpha = 0$, then

$$F(0,1;\gamma,\delta;X_3,Y_3) = (b\gamma^2 + d\delta^2)X_3^2 + (d\gamma^2 + e\delta^2)Y_3^2.$$

Hence π_{12} is quasi-finite at $\bigl([0,1],[\gamma,\delta],[\alpha_3,\gamma_3]\bigr)$ unless

$$b\gamma^2 + d\delta^2 = d\gamma^2 + e\delta^2 = 0.$$

Since $(\gamma,\delta) \ne (0,0)$, this is possible if and only if $be = d^2$.

Similarly, if $\beta = 0$, we look at

$$F(1,0;\gamma,\delta;X_3,Y_3) = (a\gamma^2 + b\delta^2)X_3^2 + (b\gamma^2 + d\delta^2)Y_3^2.$$

Thus σ_3 is well-defined at $\bigl([1,0],[\gamma,\delta],[\alpha_3,\gamma_3]\bigr)$ unless

$$a\gamma^2 + b\delta^2 = b\gamma^2 + d\delta^2 = 0.$$

Since $(\gamma,\delta) \ne (0,0)$, this is possible if and only if $ad = b^2$. This completes the proof that π_{12} is quasi-finite if and only if $be \ne d^2$ and $ad \ne b^2$. $\square$

8. Connected Fibral Components and the Cage for MK3 Surfaces

For this section we let $\mathcal{W}$ be an MK3-surface, as described in Definition 7.2, defined over a finite field $\mathbb{F}_q$. We note that the $\mathfrak{S}_3$-symmetry of $\mathcal{W}$ implies that for any $t \in \mathbb{P}^1(\mathbb{F}_q)$, the three fibers $\mathcal{W}_t^{(1)}(\mathbb{F}_q)$, $\mathcal{W}_t^{(2)}(\mathbb{F}_q)$ and $\mathcal{W}_t^{(3)}(\mathbb{F}_q)$ have the same orbit structure, so in particular

$$\begin{aligned} &\mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) \text{ for some } i \in \{1,2,3\} \\ &\qquad\qquad \iff \quad \mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) \text{ for all } i \in \{1,2,3\}. \end{aligned}$$

Thus the $\mathcal{G}$-connected fibers in $\mathcal{W}(\mathbb{F}_q)$ are determined by the projection to $\mathbb{P}^1(\mathbb{F}_q)$ of $\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr)$ onto any of its coordinates. We denote this set by

$$\pi\,\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) = \Bigl\{ t \in \mathbb{P}^1(\mathbb{F}_q) : \mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) \Bigr\}.$$

Then we have the useful characterization (for MK3-surfaces):

$$P \in \mathsf{Cage}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr) \iff \text{some coordinate of } P \text{ is in } \pi\,\mathsf{ConnFib}\bigl(\mathcal{W}(\mathbb{F}_q)\bigr).$$

9. A One Parameter Family of MK3 Surfaces

In the next few sections we study an interesting 1-parameter family of MK3-surfaces. We assume throughout that K is a field with $\mathrm{char}(K) \neq 2$.

Definition 9.1. For $k \in K^*$ we define $\mathcal{W}_k$ to be the MK3-surface

$$\mathcal{W}_k : x^2 + y^2 + z^2 + x^2y^2z^2 + kxyz = 0.$$

Remark 9.2. In the notation of Proposition 7.5, the $(2,2,2)$-form defining $\mathcal{W}_k$ has $(a,b,c,d,e) = (1,0,k,1,0)$. In particular, we have

$$be = 0 \neq 1^2 = d^2 \quad \text{and} \quad ad = 1 \neq 0^2 = b^2,$$

so Proposition 7.5(b) tells us that $\mathcal{W}_k$ is non-degenerate.

Remark 9.3. Let $\zeta \in K$ be an element satisfying $\zeta^4 = 1$. Then there is a K-isomorphism

$$\mathcal{W}_k \longrightarrow \mathcal{W}_{\zeta^3 k}, \quad (x,y,z) \longmapsto (\zeta x, \zeta y, \zeta z). \tag{17}$$

So we always have an identification $\mathcal{W}_k(K) \cong \mathcal{W}_{-k}(K)$, and if K contains $i = \sqrt{-1}$, then there are further identifications $\mathcal{W}_k(K) \cong \mathcal{W}_{\pm ik}(K)$.

Remark 9.4. The three involutions (6) on $\mathcal{W}_k$ are given explicitly by

$$\begin{aligned}
\sigma_1(x,y,z) &= \left(-\frac{kyz}{1+y^2z^2} - x, y, z\right),\\
\sigma_2(x,y,z) &= \left(x, -\frac{kxz}{1+x^2z^2} - y, z\right),\\
\sigma_3(x,y,z) &= \left(x, y, -\frac{kxy}{1+x^2y^2} - z\right).
\end{aligned}$$

We recall from Section 7 that $\mathcal{G}^\circ$ is the group $(\boldsymbol{\mu}_2^3)_1 \rtimes \mathfrak{S}_3$ of order 24 sitting in $\mathrm{Aut}(\mathcal{W}_k)$ composed of sign changes and coordinate permutations, that $\mathcal{G}^\sigma$ is the normal subgroup of $\mathrm{Aut}(\mathcal{W}_k)$ generated by $\sigma_1, \sigma_2, \sigma_3$, and that $\mathcal{G} = \mathcal{G}^\circ\mathcal{G}^\sigma$ is the subgroup of $\mathrm{Aut}(\mathcal{W}_k)$ generated by $\mathcal{G}^\circ$ and $\mathcal{G}^\sigma$.

Proposition 9.5. *Let $k \in K^*$. The set of singular points of $\mathcal{W}_k$ always contains the 4 points*

$$\bigl\{(0,0,0),\ (0,\infty,\infty),\ (\infty,0,\infty),\ (\infty,\infty,0)\bigr\}. \tag{18}$$

The point $(0,0,0)$ is fixed by $\mathcal{G}$, and the other 3 singular points form a $\mathcal{G}$-orbit.[6] *If $k \notin \{\pm 4, \pm 4i\}$, then the set (18) is the full set of singular points of $\mathcal{W}_k$.*

[6]If we also allow the δ-inversion involutions described in Remark 9.6, then the 4 singular points form a single orbit.

For $k = 4$ the set of singular points is

$$\operatorname{Sing}(\mathcal{W}_4) = \bigl\{(0,0,0),\ (0,\infty,\infty),\ (\infty,0,\infty),\ (\infty,\infty,0) \\ (1,1,-1),\ (1,-1,1),\ (-1,1,1),\ (-1,-1,-1)\bigr\}; \quad (19)$$

and for the other $k \in \{\pm 4, \pm 4i\}$, the singular points can be found using the isomorphisms described in Remark 9.3. The points in (19) with non-zero coordinates form a single $\mathcal{G}$-orbit of size 4.

Proof. We let

$$F(x,y,z) = x^2 + y^2 + z^2 + x^2y^2z^2 + kxyz \quad (20)$$

be the polynomial defining $\mathcal{W}_k$, and we use subscripts to denote partial derivatives. The singular points on this affine piece of $\mathcal{W}_k$ are the solutions to

$$F = F_x = F_y = F_z = 0. \quad (21)$$

The ideal of $\mathbb{Q}[x,y,z,k]$ generated by the four polynomials in (21) contains the following polynomials:[7]

$x^2 - y^2$	$x(x^4-1)$	$x(2^4x^2 - k^2)$	$x(k^4 - 2^8)$
$x^2 - z^2$	$y(y^4-1)$	$y(2^4y^2 - k^2)$	$y(k^4 - 2^8)$
$y^2 - z^2$	$z(z^4-1)$	$z(2^4z^2 - k^2)$	$z(k^4 - 2^8)$

(22)

The point $(0,0,0)$ is always singular. Since (22) says that singular points satisfy $x^2 = y^2 = z^2$, any other singular point (x,y,z) necessarily has $xyz \ne 0$, and then (22) forces

$$k^4 = 2^8, \quad 2^4x^2 = 2^4y^2 = 2^4z^2 = k^2, \quad \text{and} \quad x^4 = y^4 = z^4 = 1.$$

From $k^4 = 2^8$, we see that $k \in \{\pm 4, \pm 4i\}$; and from $x^4 = y^4 = z^4 = 1$, we see that $x,y,z \in \{\pm 1, \pm i\}$. For each of these 4 possible values of k, it can be directly checked that the points satisfying $F = F_x = F_y = F_z$ are those given in the table in the statement of the proposition.

It remains to check the points on the complement in $(\mathbb{P}^1)^3$ of the affine piece. To do that, we use the fact that $(0,0,0)$ is the only singular point of the affine piece of $\mathcal{W}_k$ that has a coordinate mapped to ∞ under the $\delta_{\alpha,\beta,\gamma}$ inversion maps described in Remark 9.6. By symmetry, it suffices to check points P of the following forms, where y and z are

[7] Indeed, this is true in the ring $\mathbb{Z}[2^{-1}, x, y, z, k]$.

non-zero:

P	Singular?	Why?
(∞, y, z)	No	$\delta_{-1,-1,1}(P) = (0, y^{-1}, z)$
(∞, ∞, z)	No	$\delta_{-1,-1,1}(P) = (0, 0, z)$
$(\infty, y, 0)$	No	$\delta_{-1,-1,1}(P) = (0, y^{-1}, 0)$
$(\infty, \infty, 0)$	Yes	$\delta_{-1,-1,1}(P) = (0, 0, 0)$
$(\infty, 0, 0)$	–	$\notin \mathcal{W}_k$
(∞, ∞, ∞)	–	$\notin \mathcal{W}_k$

□

Remark 9.6 (MK3-Surfaces with Extra Involutions)**.** The family of MK3-surfaces $\mathcal{W}_k$ admit additional involutions in which two of x, y, z are replaced by their multiplicative inverses.[8] Thus analogously to (14) and (15), we can define another action of $(\boldsymbol{\mu}_2^3)_1$ on $(\mathbb{P}^1)^3$ via the formula

$$\delta_{\alpha,\beta,\gamma}(x, y, z) = (x^\alpha, y^\beta, z^\gamma), \quad \text{where } (\alpha, \beta, \gamma) \in (\boldsymbol{\mu}_2^3)_1. \tag{23}$$

We observe that the δ and ϵ actions commute (since $(-1)^{-1} = -1$), so we obtain an embedding

$$\hat{\mathcal{G}}^\circ := \underbrace{\bigl((\boldsymbol{\mu}_2^3)_1 \times (\boldsymbol{\mu}_2^3)_1\bigr) \rtimes \mathfrak{S}_3}_{\text{We view this as a subgroup of } \operatorname{Aut}((\mathbb{P}^1)^3).} \longrightarrow \operatorname{Aut}(\mathcal{W}_k).$$

Since the classical Markoff equation (4) and general MK3-surfaces (16) do not admit these extra automorphisms, we will not include them when constructing orbits in $\mathcal{W}_k$. So for example, the finite orbits and $\mathcal{G}^\circ$-generators in $\mathcal{W}_k(\mathbb{C})$ that we list in Table 3 are $\mathcal{G}$-orbits, as are the finite field orbits in $\mathcal{W}_k(\mathbb{F}_p)$ in Tables 5–8. There would be some collapsing of generators and merging of orbits if we also used the δ-automorphisms. However, the existence of these extra automorphisms can aid in studying the geometry of $\mathcal{W}_k$, as will be illustrated in the proof of Proposition 9.7.

More generally, Proposition 7.5 says that MK3-surfaces $\mathcal{W}_{a,b,c,d,e}$ are described by $(2,2,2)$-forms $F_{a,b,c,d,e}(x, y, z)$ that depend on 5 homogeneous parameters $[a, b, c, d, e]$. Then the formula

$$F_{a,b,c,d,e}(x, y, z) - F_{a,b,c,d,e}(x^{-1}, y^{-1}, z)x^2y^2 = \Bigl((a-d)z^2 + (b-e)\Bigr)(x^2y^2 - 1),$$

combined with the x, y, z symmetry of $F_{a,b,c,d,e}$, imply that

$$\delta_{\alpha,\beta,\gamma} \in \operatorname{Aut}(\mathcal{W}_{a,b,c,d,e}) \iff a = d \text{ and } b = e.$$

[8] Note that we're really working in $\mathbb{P}^1$, so we formally set $0^{-1} = \infty$ and $\infty^{-1} = 0$.

Thus $\mathcal{W}_k = \mathcal{W}_{1,0,k,1,0}$ corresponds to $a = d = 1$ and $b = e = 0$.

Proposition 9.7. *Let K be a field with $\operatorname{char}(K) \ne 2$, let $k \in K^*$, and let $\xi \in \mathbb{P}^1(K)$. Then the fiber $\mathcal{W}^{(1)}_{k,\xi}$ is singular if and only if*

$$\xi = 0 \quad \textit{or} \quad \xi = \infty \quad \textit{or} \quad k = \pm 2(\xi \pm \xi^{-1}).$$

The singular points on the singular fibers are as follows:

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{k,0}\bigr) = \bigl\{(0,0,0),\,(0,\infty,\infty)\bigr\},$$
$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{k,\infty}\bigr) = \bigl\{(\infty,\infty,0),\,(\infty,0,\infty)\bigr\},$$

and for all $\xi \notin \{0,\infty\}$ and for all $u \in \{\pm 1\}$ and all $v \in \{\pm 1, \pm i\}$,

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{u(\xi+v\xi^{-1}),\xi}\bigr) = \bigl\{(\xi, v, -uv^3),\,(\xi,-v,uv^3)\bigr\}.$$

By symmetry, analogous statements are true for $\mathcal{W}^{(2)}_{k,\xi}$ and $\mathcal{W}^{(3)}_{k,\xi}$.

Remark 9.8. Let $\mathcal{W}^{(i)}_{k,\xi}$ be a fiber of $\mathcal{W}_k$. Then each of the involutions $\sigma_1, \sigma_2, \sigma_3$ and each of the automorphisms in $\mathcal{G}^\circ$ defines an isomorphism from $\mathcal{W}^{(i)}_{k,\xi}$ to some other (or possibly the same) fiber of $\mathcal{W}_k$. It follows that the singular points on a fiber are mapped to singular points on a fiber. Hence the set

$$\bigcup_{i=1}^{3} \bigcup_{\xi\in\mathbb{P}^1} \operatorname{Sing}(\mathcal{W}^{(i)}_{k,\xi})$$

of fibral singular points is a finite subset of $\mathcal{W}_k$ that is $\mathcal{G}$-invariant, so it breaks up into a finite number of finite $\mathcal{G}$-orbits. If $\xi \ne 0, \infty$ and $\xi^4 \ne 1$, then it will be a $\mathcal{G}$-orbit of size 24; cf. Table 3.

Proof of Proposition 9.7. As in the proof of Proposition 9.5, we let F be the polynomial (20) defining $\mathcal{W}_k$, and we use subscripts to denote partial derivatives. The fiber $\mathcal{W}^{(1)}_{k,\xi}$ is singular if and only if the simultaneous equations

$$F(\xi,y,z) = F_y(\xi,y,z) = F_z(\xi,y,z) = 0 \tag{24}$$

have a solution. We compute

$$\begin{aligned}\operatorname{Res}_y\Bigl(\operatorname{Res}_z(F,F_z), \operatorname{Res}_z(F_y,F_z)\Bigr) &= 2^{12}\cdot k^8 \cdot x^{26}\cdot(2x^2-kx-2)^2\\ &\quad\cdot(2x^2-kx+2)^2\cdot(2x^2+kx-2)^2\cdot(2x^2+kx+2)^2.\end{aligned}$$

We first consider the case that $\xi = 0$. Then (24) forces $y = z = 0$, so the only affine singular point is $(0,0,0)$. Using the inversion automorphism fixing the x-coordinate that is described in Remark 9.6, there is an additional singular point $(0,\infty,\infty)$, so we find that

$$\operatorname{Sing}(\mathcal{W}^{(1)}_{k,0}) = \bigl\{(0,0,0),\,(0,\infty,\infty)\bigr\}.$$

And similarly, using the inversion automorphisms in Remark 9.6 that replace the x-coordinate with x^{-1}, we see that

$$\operatorname{Sing}(\mathcal{W}^{(1)}_{k,\infty}) = \bigl\{(\infty,\infty,0),\,(\infty,0,\infty)\bigr).$$

We now assume that $\xi \ne 0,\infty$. Then our assumptions that $\operatorname{char}(K) \ne 2$ and $\mathcal{W}^{(1)}_{k,x_0}$ is singular imply that ξ is a root of one of the polynomials $2x^2 \pm kx \pm 2$. We will consider the case that

$$2\xi^2 + k\xi + 2 = 0,$$

and leave the similar computation for the other three cases to the reader. Thus we assume that

$$k = -2(\xi+\xi^{-1}) \quad\text{and}\quad \mathcal{W}^{(1)}_{k,\xi} \text{ is singular.}$$

Substituting the expression for k into (24), we find that (y_0,z_0) is a singular point on the fiber $\mathcal{W}^{(1)}_{k,\xi}$ if and only if (y_0,z_0) satisfy

$$\begin{aligned}(y^2z^2-2yz+1)\xi^2 - 2yz + y^2 + z^2 &= 0,\\ (yz^2-z)\xi^2 - z + y &= 0,\\ (y^2z-y)\xi^2 - y + z &= 0.\end{aligned}$$

Eliminating x or y or z from these three equations, we find that (y_0,z_0) satisfy

$$y^2-1 = z^2-1 = (y-z)(yz-1) = 0,$$

and these equations have two solutions,

$$(y_0,z_0) = (1,1) \quad\text{and}\quad (y_0,z_0) = (-1,-1).$$

Finally, we substitute $k = -2(\xi+\xi^{-1})$ and $(x,y,z) = (\xi,\pm1,\pm1)$ into (24) and verify that F, F_y, and F_z vanish. This proves that

$$\operatorname{Sing}\bigl(\mathcal{W}^{(1)}_{-2(\xi+\xi^{-1}),\xi}\bigr) = \bigl\{(\xi,1,1),\,(\xi,-1,-1)\bigr\} \quad \text{for all } \xi \ne 0,\infty,$$

which completes the proof of Proposition 9.7. □

Remark 9.9. For a general TIK3-surface, the three projection maps $\mathcal{W} \to \mathbb{P}^1$ give $\mathcal{W}$ three different structures as a surface fibered by genus 1 curves, and the corresponding Jacobian variety has a section of infinite order whose translation action on $\mathcal{W}$ is the σ_i associated to the projection. For MK3-surfaces, the $\mathfrak{S}_3$-symmetry implies that the three structures are the same. Using the explicit description of the singular points on $\mathcal{W}_k$ in Proposition 9.5 and the singular fibers of $\mathcal{W}_k$

in Proposition 9.7, one could compute a Néron model for $\mathcal{W}_k \to \mathbb{P}^1$ and compute the canonical height of the point on its Jacobian, but we will not do this computation in the present article.

Proposition 9.10. *Let $\mathcal{W}_k$ be the MK3-surface given in Definition 9.1, let F be the associated polynomial, let $y_0, z_0 \in \mathbb{P}^1$, and let $\mathcal{C}^{(1)}_{y_0,z_0}$ be the curve associated to F as given in Definition 6.4. If $\mathcal{C}^{(1)}_{y_0,z_0}$ is singular, then one of the following is true:*

$$y_0 \text{ or } z_0 = 0 \text{ or } \infty, \quad y_0^2 = z_0^2, \quad y_0^2 z_0^2 = 1, \quad y_0 \text{ or } z_0 = \frac{\pm k \pm \sqrt{k^2 \pm 16}}{4}.$$

By symmetry, analogous statements are true for $\mathcal{C}^{(2)}_{x_0,z_0}$ and $\mathcal{C}^{(3)}_{x_0,y_0}$.

Corollary 9.11. *Let $k \in \mathbb{F}_q^*$. Then*

$$\#\left\{(x_0, y_0, z_0) \in \mathcal{W}_k(\mathbb{F}_q) : \begin{array}{c} \textit{one or more of } \mathcal{C}^{(1)}_{y_0,z_0}, \\ \mathcal{C}^{(2)}_{x_0,z_0},\ \mathcal{C}^{(3)}_{x_0,y_0} \textit{ is singular} \end{array}\right\} \le 144q.$$

Proof of Proposition 9.10. To ease notation, we let $b = y_0$ and $c = z_0$. An affine piece of the curve $\mathcal{C}^{(1)}_{b,c}$ is given by the equations

$$F(x, b, z) = F(x, y, c) = 0.$$

Hence a point $(x, y, z) \in \mathcal{C}^{(1)}_{b,c}$ is a singular point if and only if

$$\operatorname{rank} \begin{bmatrix} F_x(x,b,z) & 0 & F_z(x,b,z) \\ F_x(x,y,c) & F_y(x,y,c) & 0 \end{bmatrix} \le 1.$$

The rank condition and a bit of algebra yields three cases, which we consider in turn.

Case 1: $\boldsymbol{F_z(x,b,z) = F_y(x,y,c) = 0}$. In this case we are looking for values of b, c, k such that the equations

$$F(x, b, z) = F(x, y, c) = F_z(x, b, z) = F_y(x, y, c) = 0$$

have a solution $(x, y, z) \in \mathbb{A}^3$. Eliminating x, y, z from these four equations gives the equation

$$(b^2 - c^2)(b^2 c^2 - 1) = 0.$$

Hence if there is a singular point, then $c = \pm b^{\pm 1}$.

Case 2: $\boldsymbol{F_x(x,b,z) = F_z(x,b,z) = 0}$. In this case, which is a version of Proposition 9.7, we are looking for values of b, c, k such that the equations

$$F(x, b, z) = F(x, y, c) = F_x(x, b, z) = F_z(x, b, z) = 0$$

have a solution $(x, y, z) \in \mathbb{A}^3$. Eliminating x, y, z from these four equations gives the equation

$$b^2(2b^2 - bk - 2)(2b^2 - bk + 2)(2b^2 + bk - 2)(2b^2 + bk + 2) = 0.$$

Hence if there is a singular point, then

$$b = 0 \quad \text{or} \quad b = \frac{\pm k \pm \sqrt{k^2 \pm 16}}{4}.$$

Case 3: $\boldsymbol{F_x(x,y,c) = F_y(x,y,c) = 0}$. By symmetry, this is the same as Case 2 with $y \leftrightarrow z$ and $b \leftrightarrow c$. □

Proof of Corollary 9.11. It suffices to bound the number of $(y_0, z_0) \in \mathbb{P}^1(\mathbb{F}_q)$ such that $\mathcal{C}^{(1)}_{y_0,z_0}$ is singular, and then multiply by 3 for the xyz-symmetry and also multiply by 2 because each (y_0, z_0) may yield 2 points on $\mathcal{W}_k$. (This includes some duplicates, so some improvement is possible.)

According to Proposition 9.10, the singular cases are included in the following table, where again we do not worry that some points appear more than once:

(y_0, z_0)	# with $\mathcal{C}^{(1)}_{y_0,z_0}$ singular
y_0 or $z_0 = 0$ or ∞	$\le 4q$
$y_0^2 = z_0^2 \ne 0$ or ∞	$\le 2(q-1)$
$y_0^2 z_0^2 = 1$	$\le 2(q-1)$
y_0 or $z_0 = \frac{\pm k \pm \sqrt{k^2 \pm 16}}{4}$	$\le 16q$

Hence there are at most $24q$ pairs (y_0, z_0), and as noted earlier, this must be multiplied by 6 to account for the other cases. □

10. Finite Orbits in $\mathcal{W}_k(\mathbb{C})$

Table 3 describes finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$. We do not claim that this is the complete list of possibilities. However, we note that the varied nature of the finite orbits in the 1-parameter family $\mathcal{W}_k$ suggests that any description of finite orbits over $\mathbb{C}$ on general TIK3-surfaces, or even on MK3-surfaces, is likely to be quite complicated.

Most of the orbits in Table 3 were unearthed by examining small orbits in $\mathcal{W}_k(\mathbb{F}_p)$ that appear in Tables 5–8 and looking at specific properties of the points in the orbits. We explain the process for a number of examples.

Question 10.1 (Uniform Boundedness Question)**.** For each $k \in \mathbb{C}$, we know from [10] that there are only finitely many finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$. Is there a bound that is independent of k for the largest such orbit? More generally, is there such a bound for finite orbits in $\mathcal{W}(\mathbb{C})$ as $\mathcal{W}$ runs over all MK3-surfaces? And even more generally, how about for all TIK3-surfaces, although in this case we look at orbits for the group generated by the three involutions $\sigma_1, \sigma_2, \sigma_3$?

Remark 10.2. We mention that if we consider $\langle\sigma_1, \sigma_2, \sigma_3\rangle$-orbits, then the orbit of size 144 in Remark 10.6 consist of 12 orbits of size 12, the orbit of size 160 in Remark 10.7 consist of 4 orbits of size 40, and the orbit of size 288 described in Remark 10.8 consist of 12 orbits of size 24. These provide lower bounds for the putative uniform bounds discussed in Questions 1.3 and10.1.

Definition 10.3 (Trivial Orbits)**.** As noted in Proposition 9.5, the four singular points in $\mathcal{W}_k$ form two $\mathcal{G}$-orbits, namely the fixed point

$$\bigl\{(0,0,0)\bigr\}$$

and the orbit of size 3,

$$\bigl\{(0,\infty,\infty),\ (\infty,0,\infty),\ (\infty,\infty,0)\bigr\}.$$

We will call these orbits the *trivial orbits* in $\mathcal{W}_k$, and as such, we have not included them in Tables 5–8.

Remark 10.4 (One-dimensional families of finite orbits in $\mathcal{W}_k(\mathbb{C})$)**.** Table 3 contains several examples of one-dimensional families of finite orbits in $\mathcal{W}_k(\mathbb{C})$, and indeed, these families are defined over $\mathbb{Q}$ or $\mathbb{Q}(i)$. Ignoring the trivial orbits described in Definition 10.3, we have the following examples:

- **Size 24:** There is a $k \in \mathbb{Q}(t)$ such that $\mathcal{W}_k\bigl(\mathbb{Q}(t)\bigr)$ has a $\mathcal{G}$-orbit of size 24.
- **Size 48:** The set $\mathcal{W}_k\bigl(\mathbb{Q}(i)\bigr)$ has a $\mathcal{G}$-orbit of size 48
- **Size 192:** There is a $k \in \mathbb{Q}(t)$ such that $\mathcal{W}_k\bigl(\mathbb{Q}(t)\bigr)$ has a $\mathcal{G}$-orbit of size 192.
- **Size 288:** There is a curve $C/\mathbb{Q}$ of genus 9 and an element $k \in \mathbb{Q}(C)$ in the function field of C so that $\mathcal{W}_k\bigl(\mathbb{Q}(C)\bigr)$ has a $\mathcal{G}$-orbit of size 288.

Remark 10.5 (Orbits of Size 64)**.** We describe the derivation of the orbit of size 64 in Table 3. Experimentally in Tables 5–8 we see orbits of size 64 in $\mathcal{W}_k(\mathbb{F}_p)$ for various values of p and k, but the relation between p and k is not clear. Examining the actual orbits in several of these cases, we found that there was a single point in $\mathcal{W}_k(\mathbb{F}_p)$ of the form (β,β,β), and that the point $(\beta,\beta,1)$ also appeared in $\mathcal{W}_k(\mathbb{F}_p)$. We next computed

$$(\beta,\beta,\beta) \in \mathcal{W}_k \quad\Longleftrightarrow\quad \beta^6 + k\beta^3 + 3\beta^2 = 0,$$

$$(\beta,\beta,1)\in\mathcal{W}_k \quad\Longleftrightarrow\quad \beta^4+(k+2)\beta^2+1=0.$$

Eliminating k and the trivial solutions $\beta\in\{0,1\}$ gives the equation[9]

$$\beta^3+\beta^2+\beta-1=0.$$

This gives $k=-(\beta+\beta^{-1})^2$. It is then an exercise to compute the $\mathcal{G}$-orbit of (β,β,β). It turns out to be the union of the $\mathcal{G}^\circ$ orbits of the following five points:

Point	(β,β,β)	$(\beta,\frac{1}{\beta},\frac{1}{\beta})$	$(\beta,\beta,1)$	$(\frac{1}{\beta},\frac{1}{\beta},1)$	$(\beta,\frac{1}{\beta},1)$
Size of $\mathcal{G}^\circ$-orbit	4	12	12	12	24

Remark 10.6 (Orbits of Size 144)**.** The orbits of size 144 in Tables 5–8 tend to feature points of the form $(\alpha,\beta,1)$ and $(\alpha,\beta,-\beta)$ that satisfy

$$\sigma_1(\alpha,\beta,-\beta)=(\alpha,\beta,-\beta)\quad\text{and}\quad\sigma_3(\alpha,\beta,-\beta)=(\alpha,\beta,1).$$

We assume that $\alpha,\beta\notin\{0,\infty\}$ and that $\beta\neq-1$, and then we obtain four conditions on k,α,β:

$$\begin{aligned}
(\alpha,\beta,1)\in\mathcal{W}_k &\quad\Longleftrightarrow\quad k=-(\alpha+\alpha^{-1})(\beta+\beta^{-1}),\\
(\alpha,\beta,-\beta)\in\mathcal{W}_k &\quad\Longleftrightarrow\quad \alpha\beta^2k=\alpha^2(\beta^4+1)+2\beta^2,\\
\sigma_1(\alpha,\beta,-\beta)=(\alpha,\beta,-\beta) &\quad\Longleftrightarrow\quad \alpha^2\beta^2(\beta^4+1)=2\beta^2,\\
\sigma_3(\alpha,\beta,-\beta)=(\alpha,\beta,1) &\quad\Longleftrightarrow\quad (\beta^2-\beta+1)\alpha^2+\beta=0.
\end{aligned}$$

The ideal in $\mathbb{Z}[\alpha,\beta,k]$ generated by these four relations is also generated (according to Magma) by the three relations

$$\alpha^4+4\alpha^2-1=0,\quad k=4\alpha(\alpha^2+4),\quad \beta^2+(\alpha^2+3)\beta+1=0.$$

(We also note that since $\alpha\neq0$, we can replace the formula for k by $k=4\alpha^{-1}$.)

Remark 10.7 (Orbits of Size 160)**.** The orbits of size 160 in Tables 5–8 tend to include a single point of the form (β,β,β) having the property that

$$\sigma_1\circ\sigma_3(\beta,\beta,\beta)=(1,\beta,*). \tag{25}$$

The assumption that $(\beta,\beta,\beta)\in\mathcal{W}_k$ gives $k=-(3+\beta^4)/\beta$, and then computing (25) explicitly gives

$$\sigma_1\circ\sigma_3(\beta,\beta,\beta)=\left(\frac{\beta^9+2\beta^5+5\beta}{\beta^8+6\beta^4+1},\beta,\frac{2\beta}{\beta^4+1}\right).$$

[9] We note that $\beta=0$ gives the contradiction $1=0$, while $\beta=1$ yields $k=-4$ and an orbit with fewer than 64 elements.

Setting the first coordinate to 1 and discarding the trivial solution $\beta = 1$ yields the condition

$$\beta^8 + 2\beta^4 - 4\beta^3 - 4\beta^2 - 4\beta + 1.$$

Setting $\gamma = 2\beta/(\beta^4+1)$ for convenience, we find that the union of the $\mathcal{G}^\circ$-orbits of the following points is an orbit of size 160.

Point	Size of $\mathcal{G}^\circ$-orbit
(β,β,β)	4
$(\beta^{-1},\beta^{-1},\beta)$	12
(β,β,γ)	12
$(\beta^{-1},\beta^{-1},\gamma)$	12
$(\beta,\beta^{-1},\gamma^{-1})$	24

Point	Size of $\mathcal{G}^\circ$-orbit
$(1,\beta,\gamma)$	24
$(1,\beta^{-1},\gamma)$	24
$(1,\beta,\gamma^{-1})$	24
$(1,\beta^{-1},\gamma^{-1})$	24

Remark 10.8 (Orbits of Size 288)**.** There is an orbit of size 288 in $\mathcal{W}_{11}(\mathbb{F}_{47})$ whose points have coordinates in the following set of values:

	t	$-t$	t^{-1}	$-t^{-1}$
α	3	44	16	31
β	6	41	8	39
γ	11	36	30	17
δ	15	32	22	25

In particular, we find that

$$\sigma_3(3,6,11) = (3,6,15) \quad \text{in } \mathcal{W}_{11}(\mathbb{F}_{47}).$$

If we now treat α,β,γ as indeterminates and want to require that

$$(\alpha,\beta,\gamma) \in \mathcal{W}_k \quad \text{and that} \quad \sigma_3(\alpha,\beta,\gamma) = (\alpha,\beta,\delta),$$

then we find that k and δ are given by the formulas

$$k = -\frac{\alpha^2+\beta^2+\gamma^2+\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma}, \tag{26}$$

$$\delta = \frac{\alpha^2+\beta^2}{\gamma(\alpha^2\beta^2+1)}. \tag{27}$$

Let $P_1 = (3,6,11) \in \mathcal{W}_{11}(\mathbb{F}_{47})$. Then the $\mathcal{G}$-orbit of P_1 has size 288, while the sub-orbit for $\mathcal{G}^\sigma = \langle \sigma_1,\sigma_2,\sigma_3\rangle$ has size 24 and is described in detail in Table 1. We observe that the subgroup of $\mathcal{G}^\circ$ leaving the orbit $\mathcal{G}^\sigma \cdot P_1$ invariant is

$$\operatorname{Stab}_{\mathcal{G}^\circ}(\mathcal{G}^\sigma \cdot P_1) = \{e,\lambda\}, \quad \text{where} \quad \lambda : (x,y,z) \longmapsto (x,-z,-y).$$

Hence the full $\mathcal{G}$-orbit of $P_1 \in \mathcal{W}_{11}(\mathbb{F}_{47})$ has order

$$\#\mathcal{G}\cdot P_1 = \Bigl(\#\mathcal{G}^\sigma \cdot P_1\Bigr)\cdot\left(\frac{\#\mathcal{G}^\circ}{\#\operatorname{Stab}_{\mathcal{G}^\circ}(\mathcal{G}^\sigma\cdot P_1)}\right) = 24\cdot\frac{24}{2} = 288.$$

Looking at Table 1, we find many relations in $\mathcal{W}_{11}(\mathbb{F}_{47})$, including for example[10]

$$\delta = \sigma_1(\alpha,\beta,\gamma)[1]^{-1} = -\sigma_2(\alpha,\beta,\gamma)[2] = \sigma_3(\alpha,\beta,\gamma)[3], \tag{28}$$

and

$$\sigma_2 \circ \sigma_3(\alpha,\beta,\gamma) = \sigma_1 \circ \sigma_3(-\beta^{-1},-\gamma,\alpha^{-1}). \tag{29}$$

If we now view (28) and (29) as determining conditions on the indeterminate quantities α,β,γ, we find that α,β,γ must satisfy certain equations, and restricting to those equations that are satisfied by $(3,6,11)$ in $\mathbb{F}_{47}$, we find that α,β,γ must satisfy

$$\alpha^3\beta^2 - \alpha^2\beta + \alpha - \beta^3 = 0, \tag{30}$$

$$\beta^3\gamma^3 - \beta^2 + \beta\gamma - \gamma^2 = 0, \tag{31}$$

$$\alpha^3\gamma^2 + \alpha^2\gamma + \alpha + \gamma^3 = 0. \tag{32}$$

These three relations for α,β,γ define a reducible subset of $\mathbb{A}^3$, and a computation using Magma shows that this set consists of two pieces. There is a finite set of points defined by

$$3\alpha + \gamma^3 = \beta + \gamma = \gamma^4 + 3 = 0, \tag{33}$$

and there is a geometrically irreducible reduced affine curve in $\mathbb{A}^3$ given by the equations

$$C = \left\{ (\alpha,\beta,\gamma) : \begin{array}{r} \alpha^2\beta - \alpha^2\gamma + \alpha\beta^2\gamma^2 - \alpha + \beta^2\gamma - \beta\gamma^2 = 0 \\ \alpha^2\gamma^2 - \alpha\beta^2\gamma^3 + \alpha\beta + \beta\gamma^3 = 0 \\ \beta^3\gamma^3 - \beta^2 + \beta\gamma - \gamma^2 = 0 \end{array} \right\} \tag{34}$$

We discard the points (33), since the orbit collapses if $\beta = -\gamma$. A further computation shows that the affine curve C has a unique singular point at $(0,0,0)$ and that it has (geometric) genus 9.

We let I denote the ideal in $\mathbb{Q}[\alpha,\beta,\gamma]$ generated by the three polynomials (34) defining the curve C. Then for each of the points P_j in Table 1, treating α,β,γ as indeterminates and taking k and δ in $\mathbb{Q}(\alpha,\beta,\gamma)$ as specified by (26) and (27), we used Magma to check that $\sigma_i(P_j)$ is as specified in Table 1 if we work in the fraction field of the quotient ring $\mathbb{Q}[\alpha,\beta,\gamma]/I$. Hence the $\mathcal{G}^\sigma$-orbit of (α,β,γ) has size 24 when we work over this ring, and then as noted earlier, the full $\mathcal{G}$-orbit has size 288.

In summary, we have shown that there is an irreducible affine curve $C/\mathbb{Q}$ of geometric genus 9 and an element $k \in \mathbb{Q}(C)$ in the function field of C so that $\mathcal{W}_k\bigl(\mathbb{Q}(C)\bigr)$ contains twelve $\mathcal{G}^\sigma$-orbits of size 24 that combine to form one $\mathcal{G}$-orbit of size 288.

[10] We use the convenient notation $\boldsymbol{v}[j]$ to denote the jth coordinate of the vector $\boldsymbol{v}$.

P	P	$\sigma_1(P)$	$\sigma_2(P)$	$\sigma_3(P)$
P_1	(α,β,γ)	P_2	P_5	P_7
P_2	$(\delta^{-1},\beta,\gamma)$	P_1	P_3	P_{11}
P_3	$(\delta^{-1},-\alpha^{-1},\gamma)$	P_4	P_2	λP_{11}
P_4	$(-\beta^{-1},-\alpha^{-1},\gamma)$	P_3	P_6	P_{10}
P_5	$(\alpha,-\delta,\gamma)$	P_6	P_1	λP_7
P_6	$(-\beta^{-1},-\delta,\gamma)$	P_5	P_4	λP_{10}
P_7	(α,β,δ)	P_8	λP_5	P_1
P_8	$(\gamma^{-1},\beta,\delta)$	P_7	P_9	P_{12}
P_9	$(\gamma^{-1},-\alpha^{-1},\delta)$	P_{10}	P_8	λP_{12}
P_{10}	$(-\beta^{-1},-\alpha^{-1},\delta)$	P_9	λP_6	P_4
P_{11}	$(\delta^{-1},\beta,\alpha^{-1})$	P_{12}	λP_3	P_2
P_{12}	$(\gamma^{-1},\beta,\alpha^{-1})$	P_{11}	λP_9	P_8

TABLE 1. The $\mathcal{G}^\sigma$-orbit of $(\alpha,\beta,\gamma) = (3,6,11) \in \mathcal{W}_{11}(\mathbb{F}_{47})$, which we want to lift to a $\mathcal{G}^\sigma$-orbit in characteristic 0. The map $\lambda \in \mathcal{G}^\circ$ is $\lambda(x,y,z) = (x,-z,-y)$.

However, we note that there are points on the curve $C(\mathbb{C})$ for which the orbit collapses. Thus if we set δ to be equal to any of α^{-1}, $-\beta$, or γ, then the $\mathcal{G}^\circ$-orbits of the 12 points listed in Table 3 collapse pairwise, and we obtain a total $\mathcal{G}$-orbit of size 144, instead of 288. A short computation shows that if we don't allow α,β,γ to be in $\{0,\pm1,\pm i\}$, then

$$\delta = \alpha^{-1} \Longrightarrow 3\alpha^4 = -1, \quad \delta = -\beta \Longrightarrow \beta^4 = -3, \quad \delta = \gamma \Longrightarrow \gamma^4 = -3.$$

Remark 10.9 (Orbits of Size 288: A Cautionary Tale)**.** We have seen in Remark 10.8 that there is an entire 1-parameter family of orbits of size 288 in characteristic 0. However, there are also exceptional orbits of size 288 in finite characteristic that do not lift. For example, we consider the orbit of size 288 in $\mathcal{W}_{11}(\mathbb{F}_{53})$. This orbit contains many points of the form $(\alpha,-\alpha,1)$ and many points of the form $(0,\beta,i\beta)$. We note that an orbit containing points of this form does not fit into the family described in Remark 10.8, but this does not preclude it coming from some other characteristic 0 orbit, so we continue analyzing the present example. In particular, we see that $\mathcal{W}_{11}(\mathbb{F}_{53})$ contains the points

$$(38,-38,1) \xrightarrow{\sigma_3} (15,38,12) \xrightarrow{\sigma_2} (15,11,12) \xrightarrow{\sigma_1} (0,11,12).$$

This suggests that we should take a point $(\alpha, -\alpha, 1) \in \mathcal{W}_k$ satisfying

$$\sigma_1 \circ \sigma_2 \circ \sigma_3(\alpha, -\alpha, 1) = (0, \beta, i\beta). \tag{35}$$

The assumption that $(\alpha, -\alpha, 1) \in \mathcal{W}_k$ forces $k = (\alpha + \alpha^{-1})^2$, and the assumption that the first coordinate in (35) is 0 forces

$$\alpha^{18} - 3\alpha^{16} + 12\alpha^{14} - 16\alpha^{12} + 62\alpha^{10} - 38\alpha^8 + 44\alpha^6 - 8\alpha^4 + 9\alpha^2 + 1 = 0. \tag{36}$$

We next observe that in $\mathcal{W}_{11}(\mathbb{F}_{53})$, the orbit of $(38, -38, 1)$ has a σ_3 fixed point, specifically

$$\sigma_2 \circ \sigma_3(38, -38, 1) = (15, 11, 12) \quad \text{is fixed by } \sigma_3. \tag{37}$$

So in general we might want to impose the further condition that

$$\sigma_3 \circ \sigma_2 \circ \sigma_3(\alpha, -\alpha, 1) = \sigma_2 \circ \sigma_3(\alpha, -\alpha, 1) \tag{38}$$

to mirror the behavior in $\mathcal{W}_{11}(\mathbb{F}_{53})$. Assuming that $\alpha \neq \pm 1$, we find that (38) forces α to satisfy

$$\alpha^{12} + 2\alpha^{10} + 15\alpha^8 + 12\alpha^6 + 15\alpha^4 + 2\alpha^2 + 1 = 0. \tag{39}$$

However, the conditions (36) and (39) are incompatible in characteristic 0. Indeed, the resultant of the two polynomials in (36) and (39) is equal to $2^{80} \cdot 53^2$, so the fact that (37) is true in $\mathcal{W}_{11}(\mathbb{F}_{53})$ comes from our choice of the specific finite field $\mathbb{F}_{53}$.

Remark 10.10 (Orbits of size 256: Another Cautionary Tale)**.** There is an orbit of size 256 in $\mathcal{W}_8(\mathbb{F}_{53})$ whose points have coordinates in the following set of values:

$$\{\pm 1, \pm\alpha^{\pm 1}, \pm\beta^{\pm 1}, \pm\gamma^{\pm 1}\} \quad \text{with} \quad \alpha = 16,\ \beta = 21,\ \gamma = 39.$$

In particular, there are points

$$\begin{aligned} P_1 &= (\alpha, \alpha, \alpha) = (16, 16, 16) \in \mathcal{W}_8(\mathbb{F}_{53}), \\ P_2 &= (\alpha, \alpha, \gamma^{-1}) = (16, 16, 34) \in \mathcal{W}_8(\mathbb{F}_{53}), \\ P_3 &= (1, \alpha, \beta) = (1, 16, 21) \in \mathcal{W}_8(\mathbb{F}_{53}), \\ P_4 &= (\alpha, \beta, \gamma) = (16, 21, 39) \in \mathcal{W}_8(\mathbb{F}_{53}). \end{aligned}$$

We first note that

$$P_1 = (\alpha, \alpha, \alpha) \in \mathcal{W}_k \quad \Longrightarrow \quad k = -\frac{\alpha^4 + 3}{\alpha},$$

$$P_2 = (\alpha, \alpha, \gamma^{-1}) \in \mathcal{W}_k \quad \Longrightarrow \quad \alpha^4 + 1 - 2\alpha\gamma = 0 \quad (\text{assuming } P_2 \neq P_1), \tag{40}$$

$$P_3 = (1, \alpha, \beta) \in \mathcal{W}_k \quad \Longrightarrow \quad (\alpha^2 + 1)\beta^2 - (\alpha^4 + 3)\beta + \alpha^2 + 1 = 0, \tag{41}$$

$$P_4 = (\alpha, \beta, \gamma) \in \mathcal{W}_k \quad \Longrightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \alpha^2\beta^2\gamma^2 - (\alpha^4 + 3)\beta\gamma = 0. \tag{42}$$

This gives three relations on α, β, γ. We can use the orbit structure of $\mathcal{W}_8(\mathbb{F}_{53})$ to generate additional relations such as

$$\begin{aligned} \sigma_1(16, 16, 16) &= (39^{-1}, 16, 16) \in \mathcal{W}_8(\mathbb{F}_{53}) \\ &\Longrightarrow \quad \sigma_1(\alpha, \alpha, \alpha) = (\gamma^{-1}, \alpha, \alpha) \in \mathcal{W}_k \\ &\Longrightarrow \quad \alpha^4 - 2\alpha\gamma + 1 = 0, \qquad (43) \\ \sigma_1(16, 21, 39) &= (16, 21, 39) \in \mathcal{W}_8(\mathbb{F}_{53}) \\ &\Longrightarrow \quad \sigma_1(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma) \in \mathcal{W}_k \\ &\Longrightarrow \quad \alpha^2(\alpha^4 + 3)\beta^2 - (\alpha^4 - 1) = 0. \qquad (44) \end{aligned}$$

The five relations (40)–(44) are incompatible in characteristic 0, although they do of course have the solution $(\alpha, \beta, \gamma) = (16, 21, 39)$ in $\mathbb{F}_{53}$. More precisely, the resultant of the five polynomials (40)–(44) is $9752 = 2^3 \cdot 23 \cdot 53$, and indeed in $\mathcal{W}_2(\mathbb{F}_{23})$ we find an orbit of size 256 corresponding to $(\alpha, \beta, \gamma) = (6, 11, 18)$. So the orbits of size 256 in $\mathcal{W}_2(\mathbb{F}_{23})$ and $\mathcal{W}_8(\mathbb{F}_{53})$ do not lift to characteristic 0.

Remark 10.11 (Orbits of Size 384: A Third Cautionary Tale)**.** There is a point $P_1 = (22, 22, -23) \in \mathcal{W}_{13}(\mathbb{F}_{71})$. A direct computation shows that $\#\mathcal{G} \cdot P_1 = 384$. We let $(\alpha, \beta, \gamma, \delta) = (22, 23, 9, 44)$, and we consider the six points $P_1 \ldots, P_6 \in \mathcal{W}_{13}(\mathbb{F}_{71})$ described in Table 2. We also let $\hat{\mathcal{G}}^\circ \subset \mathrm{Aut}(\mathcal{W}_k)$ be the subgroup containing 96 automorphisms that is described in Remark 9.6. Again by direct computation[11] we find that $\mathcal{G} \cdot P_1 \subset \mathcal{W}_{13}(\mathbb{F}_{71})$ is invariant for $\hat{\mathcal{G}}^\circ$, and that it splits up into six $\hat{\mathcal{G}}^\circ$-orbits with orbit representatives $P_1, \ldots, P_6$ and orbits of size 48 or 96 as indicated in Table 2.

We now try to lift to characteristic 0, so we view $\alpha, \beta, \gamma, \delta$ as indeterminates. However, it turns out that the six conditions

$$P_i \in \mathcal{W}_k \quad \text{for} \quad i = 1, \ldots, 6$$

are inconsistent in $\mathbb{Q}[\alpha, \beta, \gamma, \delta, k]$.

11. Full Orbits in $\mathcal{W}_k(\mathbb{F}_p)$

In this section we consider total orbits in $\mathcal{W}_k(\mathbb{F}_p)$. Such orbits are necessarily finite. In Tables 5–8 we compute the orbit structure for each $3 \le p \le 79$. We use a straightforward algorithm in which we

[11] Somewhat surprisingly, for this example we find that $\mathcal{G}^\sigma \cdot P_1 = \mathcal{G} \cdot P_1 = \hat{\mathcal{G}}^\circ \mathcal{G}^\sigma \cdot P_1$ in $\mathcal{W}_{13}(\mathbb{F}_{71})$.

$\#\hat{\mathcal{G}}^\circ P$	P	P	$\sigma_1(P)$	$\sigma_2(P)$	$\sigma_3(P)$
48	P_1	$(\alpha,\alpha,-\beta)$	$(\gamma^{-1},\alpha,-\beta)$	$(\alpha,\gamma^{-1},-\beta)$	$(\alpha,\alpha,-\gamma)$
48	P_2	$(\alpha,\alpha,-\gamma)$	$(\beta^{-1},\alpha,-\gamma)$	$(\alpha,\beta^{-1},-\gamma)$	$(\alpha,\alpha,-\beta)$
48	P_3	(β,β,γ)	$(-\alpha^{-1},\beta,\gamma)$	$(\beta,-\alpha^{-1},\gamma)$	(β,β,δ)
48	P_4	(β,β,δ)	$(-1,\beta,\delta)$	$(\beta,-1,\delta)$	(β,β,γ)
96	P_5	$(\alpha,-\beta,\gamma^{-1})$	$(-\beta^{-1},-\beta,\gamma^{-1})$	$(\alpha,-\alpha^{-1},\gamma^{-1})$	$(\alpha,-\beta,\alpha)$
96	P_6	$(\beta,-\delta,1)$	$(\beta^{-1},-\delta,1)$	$(\beta,-\delta^{-1},1)$	$(\beta,-\delta,-\beta)$

TABLE 2. The $\mathcal{G}$-orbit of $(\alpha,\alpha,-\beta)=(22,22,-23)\in\mathcal{W}_{13}(\mathbb{F}_{71})$, with $\gamma=9$ and $\delta=44$. We want to lift it to a $\mathcal{G}$-orbit in characteristic 0. We note that every point in the last three columns is in the $\hat{\mathcal{G}}^\circ$-orbit of one of $P_1,\dots,P_6$.

generate a list of points on the surface, then for each point compute its full orbit and eliminate the points in its orbit from the list. Various programming tricks speed the computation, e.g., using a sorted list of hash values of the points, but there are no real computational innovations.

In view of the isomorphisms provided by Remark 9.3, for $p\equiv 3 \pmod 4$ we compute the orbit structure of $\mathcal{W}_k(\mathbb{F}_p)$ for only one of $\pm k\in\mathbb{F}_p^*$; and for $p\equiv 1 \pmod 4$, we compute the orbit structure of $\mathcal{W}_k(\mathbb{F}_p)$ for only one of $\pm k,\pm ik\in\mathbb{F}_p^*$, where $i=\sqrt{-1}\in\mathbb{F}_p$. In Tables 5–8, we have also omitted the trivial orbits of size 1 and 3 described in Definition 10.3.

Reducing the characteristic 0 orbits in Table 3 modulo p yields some of the small characteristic p orbits in Tables 5-8. In particular, Table 4 lists the characteristic p orbits of sizes 144, 160 and 288 for $p\le 79$ that come from characteristic 0.

12. Fibral Orbits in $\mathcal{W}_k(\mathbb{F}_p)$

We let

$$\mathcal{G}=\langle\sigma_1,\sigma_2,\sigma_3,\tau_{12},\tau_{13},\tau_{23},\epsilon_{12},\epsilon_{13},\epsilon_{23}\rangle\subset\operatorname{Aut}(\mathcal{W}_k).$$

For $x_0,y_0,z_0\in K$, we denote the fibers of $\mathcal{W}_k(K)$ as usual by

$$\begin{aligned}\mathcal{W}^{(1)}_{k,x_0}&=\bigl\{(x_0,y,z)\in\mathcal{W}_k(K)\bigr\},\\ \mathcal{W}^{(2)}_{k,y_0}&=\bigl\{(x,y_0,z)\in\mathcal{W}_k(K)\bigr\},\\ \mathcal{W}^{(3)}_{k,z_0}&=\bigl\{(x,y,z_0)\in\mathcal{W}_k(K)\bigr\}.\end{aligned}$$

orbit size	k	$\mathcal{G}^\circ$-generators
1	all k	$(0,0,0)$
3	all k	$(0,\infty,\infty)$
4	$k=4$	$(-1,-1,-1)$
24	$\xi^4 \ne 1$ $k=-2(\xi+\xi^{-1})$	$(\xi,1,1)$, $(\xi^{-1},1,1)$
48	all k	$(1,i,0)$, $(1,i,\infty)$
64	$\beta^3+\beta^2+\beta-1=0$ $k=-(\beta+\beta^{-1})^2$	(β,β,β), $(\beta,\beta,1)$ $(\beta^{-1},\beta^{-1},1)$ $(\beta,\beta^{-1},\beta^{-1})$ $(\beta,\beta^{-1},1)$
96	$\eta^4=-1$ $k=-2\eta^2$	$(\eta,\eta^3,0)$ (η,η^3,η^6) (η,η^2,η^5) (η,η^2,∞)
144	$\alpha^4+4\alpha^2-1=0$ $\beta^2+(\alpha^2+3)\beta+1=0$ $\beta^4+2\beta^3-2\beta^2+2\beta+1=0$ $k=4\alpha^{-1}$	$(\alpha,\beta,1)$, $(\alpha^{-1},\beta,1)$, $(\alpha,\beta^{-1},1)$, $(\alpha^{-1},\beta^{-1},1)$, $(\alpha,\beta,-\beta)$, $(\alpha^{-1},\beta^{-1},-\beta)$
160	$\beta^8+2\beta^4-4\beta^3$ $-4\beta^2-4\beta+1=0$ $\gamma=2\beta/(\beta^4+1)$ $k=-(3+\beta^4)/\beta$	(β,β,β) $(1,\beta,\gamma)$ $(\beta^{-1},\beta^{-1},\beta)$ $(1,\beta^{-1},\gamma)$ (β,β,γ) $(1,\beta,\gamma^{-1})$ $(\beta^{-1},\beta^{-1},\gamma)$ $(1,\beta^{-1},\gamma^{-1})$ $(\beta,\beta^{-1},\gamma^{-1})$
192	$\xi^8\ne 1$ $k=i(\xi^2-\xi^{-2})$	$(\xi,i\xi,0)$, $(\xi,-i\xi,1)$, $(\xi,i\xi^{-1},1)$, $(\xi,i\xi^{-1},\infty)$, $(\xi^{-1},-i\xi,1)$, $(\xi^{-1},i\xi,\infty)$, $(\xi^{-1},i\xi^{-1},0)$, $(\xi^{-1},i\xi^{-1},1)$
288 or 144*	$\alpha^2\beta-\alpha^2\gamma+\alpha\beta^2\gamma^2$ $-\alpha+\beta^2\gamma-\beta\gamma^2=0$ $\alpha^2\gamma^2-\alpha\beta^2\gamma^3+\alpha\beta+\beta\gamma^3=0$ $\beta^3\gamma^3-\beta^2+\beta\gamma-\gamma^2=0$ $\delta=\dfrac{\alpha^2+\beta^2}{\gamma(\alpha^2\beta^2+1)}$ $k=-\dfrac{\alpha^2+\beta^2+\gamma^2+\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma}$	(α,β,γ) $(\delta^{-1},\beta,\gamma)$ $(\delta^{-1},-\alpha^{-1},\gamma)$ $(-\beta^{-1},-\alpha^{-1},\gamma)$ (α,β,δ) $(\gamma^{-1},\beta,\delta)$ $(\gamma^{-1},-\alpha^{-1},\delta)$ $(-\beta^{-1},-\alpha^{-1},\delta)$ $(\alpha,-\gamma,\delta)$ $(-\beta^{-1},-\gamma,\delta)$ $(\delta^{-1},\beta,\alpha^{-1})$ $(\gamma^{-1},\beta,\alpha^{-1})$ *Orbit size 144 if $3\alpha^4=-1$ or $\beta^4=-3$ or $\gamma^4=-3$

TABLE 3. Finite $\mathcal{G}$-orbits in $\mathcal{W}_k(\mathbb{C})$, where in each case we list only one of $\mathcal{W}_{\pm k}$ and $\mathcal{W}_{\pm ik}$; cf. Remark 9.3.

The $\mathcal{G}$-fibral automorphism group of the fiber $\mathcal{W}^{(1)}_{k,x_0}$ is generated by the two involutions σ_2 and σ_3 that fix x_0, the transposition τ_{23} that swaps the y and z coordinates, and the map ϵ_{23} that changes the sign

p	k	α	β	Orbit size
11	1	4	5	144
19	8	11	4	144
29	1	4	18	144
29	11	3	2	144
31	2	2	3	144
59	9	7	21	144
71	34	21	59	144
79	6	27	63	144

Orbits of size 144: Remark 10.6

p	k	β	γ	Orbit size
19	2	6	10	160
23	5	20	19	160
31	6	22	8	160
41	1	25	35	160
41	4	31	34	160
59	8	36	38	160
67	27	11	49	160
73	18	9	16	160

Orbits of size 160: Remark 10.7

p	k	α	β	γ	Orbit size	
19	9	7	2	3	144	$\beta^4 = -3$
23	4	10	8	9	288	
43	2	28	13	14	144	$3\alpha^4 = -1$
47	11	3	6	11	288	
59	23	13	33	8	288	
61	15	4	7	18	288	
67	31	5	30	12	144	$3\alpha^4 = -1$
71	13	10	44	16	288	
79	35	36	8	59	288	
79	36	12	19	51	288	

Orbits of sizes 144 and 288: Remark 10.8

TABLE 4. $\mathcal{W}(\mathbb{F}_p)$ orbits of sizes 144, 160 and 288 in Tables 5–8 coming from $\mathcal{W}(\overline{\mathbb{Q}})$ orbits in Table 3.

of y and z; and similarly for the other fibers. Thus[12]

$$\begin{aligned}
\mathcal{G}^{(1)}_{x_0} &= \langle \sigma_2, \sigma_3, \tau_{23}, \epsilon_{23} \rangle \subset \operatorname{Aut}\big(\mathcal{W}^{(1)}_{x_0}\big),\\
\mathcal{G}^{(2)}_{y_0} &= \langle \sigma_1, \sigma_3, \tau_{13}, \epsilon_{13} \rangle \subset \operatorname{Aut}\big(\mathcal{W}^{(2)}_{y_0}\big),\\
\mathcal{G}^{(3)}_{z_0} &= \langle \sigma_1, \sigma_2, \tau_{12}, \epsilon_{12} \rangle \subset \operatorname{Aut}\big(\mathcal{W}^{(3)}_{z_0}\big).
\end{aligned}$$

We recall that since $\mathcal{W}_k$ is an MK3-surface, there is a set of points

$$\pi\,\mathsf{ConnFib}\big(\mathcal{W}_k(\mathbb{F}_q)\big) \subset \mathbb{P}^1(\mathbb{F}_q)$$

such that

[12]We have listed more generators than needed. For example, $\sigma_3 = \tau_{23} \circ \sigma_2 \circ \tau_{23}$, so $\operatorname{Aut}\big(\mathcal{W}^{(1)}_{x_0}\big) = \langle \sigma_2, \tau_{23}, \epsilon_{23} \rangle$, and similarly for the others fibers.

p	k	orbit sizes
3	1	4
5	1	$4, 48$
7	1	64
7	2	24
7	3	4
11	1	144
11	2	64
11	3	24
11	4	$4, 128$
11	5	$24, 64$
13	1	$24, 48, 192$
13	2	$24, 40, 48, 64, 120$
13	4	$4, 48, 192$
17	1	$4, 16, 24, 48^2, 64, 288$
17	2	$48, 96, 192$
17	3	$24, 48, 384$
17	6	$24, 48, 160, 192$
19	1	$24, 160$
19	2	$24, 160$
19	3	320
19	4	$4, 320$
19	5	$24, 288$
19	6	$24, 288$
19	7	432
19	8	288
19	9	$48, 64, 144^2$
23	1	$24, 448$
23	2	$256, 352$
23	3	$24, 336$
23	4	$4, 96, 288$
23	5	$24, 112, 160$
23	6	448
23	7	576
23	8	$24, 448$
23	9	608
23	10	448
23	11	$24, 384$
29	1	$40, 48, 120, 144, 192, 352$
29	2	$24, 48, 352, 672$
29	3	$24^2, 48, 1152$
29	4	$4, 48, 192^2, 288^2$
29	6	$24^2, 48, 1184$
29	8	$24, 48, 64, 96, 288, 576$
29	11	$48, 144, 192^2, 384$
31	1	$24, 800$
31	2	$24, 144, 544$
31	3	896
31	4	$4, 768$
31	5	$24, 688$
31	6	$24, 160, 256, 384$
31	7	$24, 864$
31	8	864
31	9	864
31	10	1024
31	11	1056
31	12	$24, 624$
31	13	1120
31	14	$24, 800$
31	15	1024
37	1	$36^2, 48, 72^2, 160, 192,$ $216, 288, 384$
37	2	$24, 48, 72, 216, 576, 672$
37	3	$24^2, 48, 768, 1056$
37	4	$4, 48, 192, 384, 960$
37	5	$24^2, 48, 1792$
37	8	$24, 48, 480, 1152$
37	9	$24, 48, 160, 192, 1312$
37	10	$24, 48, 1664$
37	15	$48, 160, 192^2, 288, 624$

TABLE 5. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 10.3

p	k	orbit sizes
41	1	$48, 64, 160, 1632$
41	2	$24, 40, 48, 96, 120, 192, 1536$
41	3	$24, 48, 192, 1824$
41	4	$4, 24, 40, 48, 72, 120, 160, 192^3, 216, 288, 576$
41	6	$16, 24, 48^2, 192, 1632$
41	7	$24, 48, 192, 1792$
41	8	$24, 48, 192, 1792$
41	11	$24, 48, 384, 1600$
41	12	$24^2, 48, 2160$
41	16	$48, 96, 192, 1440$
43	1	1728
43	2	$24, 48, 144, 1536$
43	3	$24, 1536$
43	4	$4, 1856$
43	5	$24, 1408$
43	6	1632
43	7	1936
43	8	1968
43	9	1760
43	10	$24, 64, 1600$
43	11	1936
43	12	$256, 1504$
43	13	$24, 1408$
43	14	1728
43	15	2032
43	16	$24, 1408$
43	17	$24, 384, 1024$
43	18	1968
43	19	$24, 1664$
43	20	$24, 256, 1408$
43	21	$24, 1728$

p	k	orbit sizes
47	1	$24, 1712$
47	2	2304
47	3	2112
47	4	$4, 1920$
47	5	$24, 2080$
47	6	2336
47	7	$64, 2016$
47	8	$24, 2080$
47	9	$24, 1776$
47	10	$24, 2080$
47	11	$64, 96, 160, 288, 1728$
47	12	$24, 64, 2016$
47	13	$24, 2080$
47	14	1984
47	15	$24, 1776$
47	16	$864, 1216$
47	17	2304
47	18	2336
47	19	$24, 1712$
47	20	$24, 2016$
47	21	$24, 1776$
47	22	2400
47	23	1984
53	1	$24^2, 48, 3456$
53	2	$48, 192, 2736$
53	3	$24^2, 48, 192, 3360$
53	4	$4, 48, 3072$
53	5	$24, 48, 64, 3168$
53	6	$24, 48, 192, 3040$
53	8	$48, 64, 192, 256, 336, 2016$
53	10	$24, 48, 192, 3072$
53	11	$24, 48, 64, 192, 288, 2688$
53	13	$24, 48, 192, 288, 2752$
53	15	$24, 48, 192, 2944$
53	17	$24, 48, 192, 3040$
53	22	$24, 48, 192^2, 2752$

TABLE 6. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 10.3

p	k	orbit sizes
59	1	3232
59	2	3328
59	3	3360
59	4	$4, 3392$
59	5	$24, 2880$
59	6	$24, 3264$
59	7	3696
59	8	$24, 160, 2848$
59	9	$144, 160, 3328$
59	10	$24, 3008$
59	11	$24, 2880$
59	12	3792
59	13	$24, 3328$
59	14	$24, 2880$
59	15	$160, 3072$
59	16	$24, 3008$
59	17	3600
59	18	3232
59	19	3632
59	20	3328
59	21	$24, 3264$
59	22	3232
59	23	$24, 96, 288, 2944$
59	24	$24, 3328$
59	25	$24, 2880$
59	26	3632
59	27	$24, 3328$
59	28	$24, 3136$
59	29	3696
61	1	$24, 48, 4224$
61	2	$24^2, 48, 4512$
61	3	$24, 48, 192, 256, 384, 3424$
61	4	$4, 48, 192, 384, 3456$
61	5	$24^2, 48, 4480$
61	7	$24, 48, 192, 4032$
61	8	$24^2, 48, 192, 4288$
61	9	$24^2, 48, 192^2, 4192$
61	10	$36^2, 48, 72, 192, 288, 3168$

p	k	orbit sizes
61	13	$48, 64, 544, 3248$
61	14	$24, 48, 352, 3904$
61	15	$24, 48, 96, 288^3, 3264$
61	19	$48, 192^2, 288, 3184$
61	20	$48, 288, 3568$
61	25	$24, 48, 192, 3936$
67	1	4320
67	2	$24, 4256$
67	3	$24, 3808$
67	4	$4, 4544$
67	5	$24, 4256$
67	6	4656
67	7	$24, 3936$
67	8	4624
67	9	$24, 4320$
67	10	$24, 3808$
67	11	4720
67	12	4352
67	13	$24, 4128$
67	14	4624
67	15	4352
67	16	$24, 3936$
67	17	4224
67	18	$24, 4256$
67	19	$24, 4256$
67	20	$24, 3936$
67	21	$24, 3808$
67	22	4720
67	23	4320
67	24	$24, 3808$
67	25	$24, 4128$
67	26	$480, 3840$
67	27	$96, 160, 288, 4080$
67	28	$288, 4528$
67	29	$24, 4320$
67	30	4624
67	31	$48, 144, 4032$
67	32	4352
67	33	$24, 3808$

TABLE 7. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 10.3

p	k	orbit sizes
71	1	5280
71	2	4768
71	3	$24, 4560$
71	4	$4, 4608$
71	5	$24, 4800$
71	6	$24, 4864$
71	7	5376
71	8	$24, 4368$
71	9	5184
71	10	4864
71	11	5280
71	12	$24, 4304$
71	13	$96, 288, 384, 4096$
71	14	$24, 4864$
71	15	5216
71	16	$24, 4800$
71	17	$24, 4864$
71	18	$24, 4672$
71	19	5184
71	20	$24, 4864$
71	21	5216
71	22	4864
71	23	$24, 4368$
71	24	4864
71	25	4768
71	26	5216
71	27	$24, 4672$
71	28	$24, 4304$
71	29	4864
71	30	$24, 4304$
71	31	4864
71	32	5216
71	33	$24, 4368$
71	34	$24, 144, 4224$
71	35	$24, 4800$
73	1	$48, 192, 5248$
73	2	$24, 48, 96, 5760$
73	3	$24, 48, 64, 5920$
73	4	$4, 24, 40, 48, 120, 160, 192, 288^2, 1920, 2976$
73	5	$24^2, 48, 6448$
73	6	$48, 192, 5376$
73	7	$24, 48, 5952$
73	9	$24^2, 48, 6288$
73	10	$48, 192, 5248$
73	12	$24, 48, 192, 5792$
73	13	$48, 192, 672, 4576$
73	15	$48, 192, 544, 4704$
73	17	$24, 48, 192, 5760$
73	18	$24^2, 48, 160, 192, 6000$
73	20	$16, 24, 48^2, 192, 5728$
73	23	$24, 48, 5856$
73	26	$24^2, 48, 6256$
73	31	$24, 48, 192, 5792$
79	1	$24, 5856$
79	2	$24, 5424$
79	3	$24, 5488$
79	4	$4, 5760$
79	5	$24, 6048$
79	6	$24, 144, 5344$
79	7	5952
79	8	5792
79	9	$24, 5488$
79	10	$24, 5984$
79	11	$24, 5984$
79	12	$24, 5424$
79	13	6432
79	14	$24, 6048$
79	15	$24, 5488$
79	16	6400
79	17	$24, 5984$
79	18	6592
79	19	6400
79	20	6048
79	21	5952
79	22	$24, 5488$
79	23	6496
79	24	6496
79	25	6048
79	26	6432
79	27	$24, 5984$
79	28	6080
79	29	5792
79	30	6496
79	31	$24, 6048$
79	32	5952
79	33	$24, 5984$
79	34	6592
79	35	$96, 288, 6112$
79	36	$24, 96, 288, 5664$
79	37	$24, 5680$
79	38	5952
79	39	$24, 64, 5616$

TABLE 8. Non-trivial orbits in $\mathcal{W}_k(\mathbb{F}_p)$; cf. Definition 10.3

$$t \in \pi\,\mathsf{ConnFib}\big(\mathcal{W}_k(\mathbb{F}_q)\big) \iff$$
$$\mathcal{W}_t^{(i)}(\mathbb{F}_q) \in \mathsf{Cage}\big(\mathcal{W}_k(\mathbb{F}_q)\big) \text{ for one (equivalently all) } i \in \{1,2,3\}.$$

Example 12.1. We consider the surface $\mathcal{W}_1$ over the finite field $\mathbb{F}_{53}$. The set $\mathcal{W}_1(\mathbb{F}_{53})$ has six $\mathcal{G}$-orbits of sizes, respectively, 1, 3, 24, 24, 48 and 3456. We compute the number of components on the various fibers, and when we do so, we find that

$$\pi\,\mathsf{ConnFib}\big(\mathcal{W}_1(\mathbb{F}_{53})\big) = \{\pm 2, \pm 4, \pm 6, \pm 13, \pm 20, \pm 24, \pm 26\}. \tag{45}$$

Next, for each t in $\pi\,\mathsf{ConnFib}\big(\mathcal{W}_1(\mathbb{F}_{53})\big)$, we would like to know which of the coordinates in $\pi\,\mathsf{ConnFib}\big(\mathcal{W}_1(\mathbb{F}_{53})\big)$ appear as the coordinate of some point in the (connected) fiber $\mathcal{W}_t^{(i)}(\mathbb{F}_{53})$. In general, if S is any set of points in $(\mathbb{P}^1)^3$, we define

$$\mathsf{Flatten}(S) = \text{the set of all coordinates of all points in } S.$$

Then we may compute the connectivity of the cage of $\mathcal{W}_1(\mathbb{F}_{53})$ using the data in the following table.

t	$\mathsf{Flatten}\big(\mathcal{W}_{1,t}^{(1)}(\mathbb{F}_{53})\big) \cap \pi\,\mathsf{ConnFib}\big(\mathcal{W}_1(\mathbb{F}_{53})\big)$
± 2	$\{\pm 6, \pm 20\}$
± 4	$\{\pm 24\}$
± 6	$\{\pm 2, \pm 20, \pm 26\}$
± 13	$\{\pm 24\}$
± 20	$\{\pm 2, \pm 6, \pm 20, \pm 26\}$
± 24	$\{\pm 4, \pm 13, \pm 24\}$
± 26	$\{\pm 6, \pm 20\}$

Thus the cage in the big component of $\mathcal{W}_1(\mathbb{F}_{53})$ is not connected. It consists of the following two pieces, which are also illustrated in Figure 2:

$$\bigcup_{t\in\{\pm 2,\pm 6,\pm 20,\pm 26\}} \bigcup_{i\in\{1,2,3\}} \mathcal{W}_{1,t}^{(i)} \quad \text{and} \quad \bigcup_{t\in\{\pm 4,\pm 13,\pm 24\}} \bigcup_{i\in\{1,2,3\}} \mathcal{W}_{1,t}^{(i)}$$

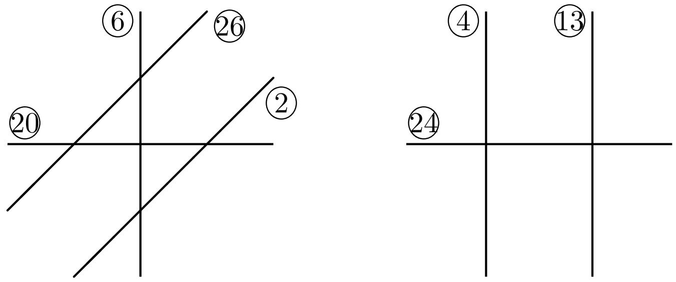

FIGURE 2. The two connected components of the cage of $\mathcal{W}_1(\mathbb{F}_{53})$, where the segment labeled ⓣ denotes the union of the six connected fibers $\cup_{i=1,2,3} \cup_{\epsilon=\pm 1} \mathcal{W}^{(i)}_{1,\epsilon t}(\mathbb{F}_{53})$

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$t_0 \backslash p$	5	7	11	13	17	19	23	29	31	37	41
∞	2	1	1	4	6	1	1	8	1	10	12
0	3	2	2	5	6	2	2	9	2	11	12
1	2	1	1	2	2	2	2	3	3	4	3
2	1	1	1	2	3	1	1	1	1	2	3
3	1	1	1	2	2	0	1	2	1	3	1
4	2	1	1	2	4	1	1	2	1	6	2
5		1	1	2	3	1	1	1	1	4	2
6		1	1	1	2	0	1	2	1	3	2
7			1	1	2	1	1	3	1	1	1
8			1	2	2	1	1	2	1	2	1
9			1	2	2	1	1	2	1	4	4
10			1	2	2	1	1	1	1	3	2
11				2	2	1	2	2	2	2	1
12				2	3	1	2	2	1	3	1
13					4	0	1	2	1	3	4
14					2	1	1	1	1	3	1
15					3	1	1	1	1	2	2
16					2	0	1	2	1	1	1
17						1	1	2	1	3	1
18						2	1	2	1	1	1
19							1	1	1	1	6
20							1	2	2	3	2
21							1	2	1	1	2
22							2	3	1	2	6
23								2	1	3	1
24								1	1	3	1
25								2	1	3	1
26								2	1	2	2
27								1	1	3	1
28								3	1	4	4
29									1	2	1
30									3	1	1
31										3	2
32										4	4
33										6	1
34										3	1
35										2	2
36										4	2
37											2
38											1
39											3
40											3

TABLE 9. # of fibral $\mathrm{Aut}(\mathcal{W}^{(i)}_{1,t_0})$-orbits in $\mathcal{W}_1(\mathbb{F}_p)$ for $i = 1, 2, 3$

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