APOLLONIAN PACKINGS: THE RISE AND FALL OF THE LOCAL TO GLOBAL CONJECTURE

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ABSTRACT. Nearly 20 years ago, a paper of Graham-Lagarias-Mallows-Wilks-Yan on the number theory of Apollonian circle packings sparked an interest in the number theory community, which was just developing tools to handle arithmetic problems involving so-called thin groups. At the time, these packings were the only naturally occurring example of such an arithmetic problem, and naturally number theorists sprang upon the opportunity to discover all their rich properties, thinness notwithstanding. In 2010, building upon conjectures of Graham-Lagarias-Mallows-Wilks-Yan, we gave evidence together with Katherine Sanden towards the Local to Global Conjecture for Apollonian circle packings, stating that in any integral packing, any large enough integer that satisfied certain congruence conditions modulo 24 must appear as a curvature in the packing. For 13 years, most everyone believed this conjecture to be true. In this article, we will explore the history of this conjecture, and its fascinating downfall after Haag-Kertzer-Rickards-Stange proved that, in fact, infinitely many integral Apollonian packings fail to abide by the local to global principle, and come with extra quadratic and quartic obstructions.

1. INTRODUCTION

Local to global style theorems and conjectures are central in many branches of number theory. Reminiscent of the idea in geometry that one can understand a (global) object such as a surface in space by studying properties near (local) fixed points on that surface, in number theory the local to global principle (sometimes referred to as the Hasse principle) is a phenomenon in which one can completely understand an arithmetic entity, such as rational points on an elliptic curve, just by studying that object over local fields – the p-adic numbers and the reals.

A well known classical example of this which is often presented in a first course in undergraduate number theory is the study of nonnegative integers which can be written as the sum of three squares:

$$n = X^2 + Y^2 + Z^2$$

where $X, Y, Z \in \mathbb{Z}$. It is not hard to see that no number which is 7 modulo 8 can be written in this way – a square can only be $0, 1$, or 4 modulo 8, and 7 is the only residue one cannot obtain from any combination of three of these. Indeed, there are no other *local* obstructions: any integer n can be written as the sum of three squares over the reals, and over $\mathbb{Q}_p$ for $p \neq 2$. What is harder to see, and was first proven by Legendre[1] in 1797, is that any odd number which is not 7 modulo 8 can, in fact, be written as a sum of three squares. This, in fact, is a local to global principle governing which odd numbers can be written

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[1]Legendre's proof was not complete, and a complete proof was given by Gauss shortly after Legendre. A more elegant proof was produced by Dirichlet several decades later. The proof of the 3-squares theorem that most closely ties in with the subject of this article comes from combining Hasse-Minkowski with a result of Aubry [2] from 1912 that if a number is represented as the sum of three rational squares, then it is represented as the sum of three integer squares.

as sums of three squares. Legendre furthermore proved what is known as Legendre's 3 Square Theorem, that a number n can be written as a sum of three squares if and only if it is not of the form $4^a(8k+7)$ for any non-negative integers a and k.

There is also a wealth of interesting examples where the local to global principle does not hold. In contrast to Legendre's 3 squares theorem, it is not in general true that if a number is represented by a quadratic form over the p-adics and reals, then it is represented over $\mathbb{Z}$: for example, while $3x^2+13y^2=1$ is solvable both over the reals and over the p-adics for every prime p, it is clearly not solvable over $\mathbb{Z}$. Indeed, Hilbert's tenth problem asks if there is an algorithm to determine whether or not any given Diophantine equation has a solution over $\mathbb{Z}$, and that problem was shown to be undecidable by work of Davis, Matiyasevich, Putnam, and Robinson [7], [25], [8], [29]. However, the Hasse-Minkowski principle states precisely this for quadratic forms with rational coefficients with $\mathbb{Z}$ replaced by $\mathbb{Q}$ – that a quadratic form with rational coefficients represents a number over $\mathbb{Q}$ if and only if it represents a number over $\mathbb{R}$ and $\mathbb{Q}_p$ for every prime p. Indeed, $3x^2+13y^2=1$ is solvable over $\mathbb{Q}$ by taking $x=\frac{2}{5}$ and $y=\frac{1}{5}$.

In yet another example of failure of a local to global principle for a well known problem, while Fermat's Last Theorem famously says that there are no nontrivial solutions to $X^n+Y^n=Z^n$ for $n>2$, one can solve the equation modulo any prime p in the case $n=3$, and, as shown by first by Dickson in [9] using Jacobi sums for n prime, and later in more generality by Schur in [30], modulo a large enough prime p for any given n (Here "large enough" depends on n).

The arithmetic object we now turn our focus to is the Apollonian circle packing. Named after Apollonius of Perga, who studied the problem of constructing a circle tangent to all of three given circles in his (now lost) work *The Tangencies* sometime in the third century BC, the first known mention of an object like an Apollonian packing is in a letter from Leibniz to Des Bosses in 1706 [23], as part of a philosophical discussion:

> *...it should not be concluded from this that an infinitely small portion of matter (such as does not exist) must be assigned to any entelechy, even if we usually rush to such conclusions by a leap. I shall use an analogy. Imagine a circle; in it draw three other circles that are the same in size and as large as possible, and in any new circle and in the space between circles again draw the three largest circles of the same size that are possible. Imagine proceeding to infinity in this way: it does not follow that there is an infinitely small circle or that there is a center having its own circle in which (contrary to the hypothesis) no other is inscribed.*

Our construction is slightly different: consider four pairwise tangent circles or lines in one of two configurations: either two tangent circles inscribed between two parallel lines (tangent at infinity) or three pairwise tangent circles inscribed inside one larger one (see Figure 1). Apollonius observed that for any three of these pairwise tangent circles or lines in either picture, there are precisely two others tangent to the other three. In the first picture, taking the three pairwise tangent circles to be the two parallel lines and one of the circles, Apollonius's theorem is obvious: the two circles tangent to these three are the two circles one can inscribe between the two lines on either side of the inside circle. Indeed, this is nice way to see Apollonius's theorem for the second configuration as well, since one can obtain the second configuration from the first by performing several circle inversions, which are tangency preserving. Given this observation, one sees that there is a unique circle one can inscribe in the region bounded by any three of these circles. Iterating this process, one gets what is known as an Apollonian circle packing.

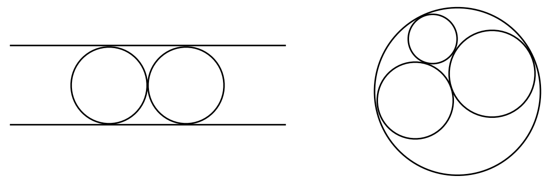


FIGURE 1. Two starting configurations of circles

As far as we know, the number theoretic aspects of this construction were discovered some two centuries later by chemistry Nobel Prize laureate Frederick Soddy, who was so taken by the problem that he wrote a poem about it [31]. Specifically, if the four circles in the starting configuration have integer curvature (e.g. if we take the two circles in the first configuration in Figure 1 to have curvature 1, with the lines having curvature 0), all of the circles in the packing will have integer curvature. Nonetheless, for a long time most works on Apollonian packings were geometrically flavored (see [1], [15], [16],[17], [20], [24], [26], [33], for example), and the first paper which delved deep into the number theory of integral Apollonian packings was [18] by Graham, Lagarias, Mallows, Wilks, and Yan in 2003.

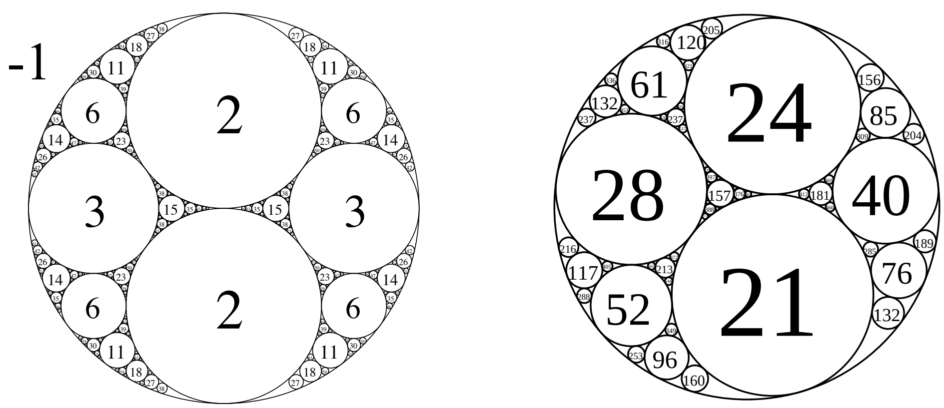


FIGURE 2. The integral Apollonian packings P_B and P_C

One of the open problems proposed in [18] had to do with a potential local to global conjecture for Apollonian packings, and indeed there is some interesting data already suggesting obstructions modulo powers of 2 and 3 in [18]. Graham-Lagarias-Mallows-Wilks-Yan had the makings of the following Local to Global Conjecture which we stated formally in joint work with Sanden in [13], backed up by convincing data.

Conjecture 1.1 (Local to Global Conjecture for Apollonian packings, [13], [18])**.** *Let P be an integral Apollonian Circle Packing and let $R(P)$ be the set of residue classes modulo* 24 *of curvatures in P. Then there exists $X_P \in \mathbb{Z}$ such that any integer $x > X_P$ whose residue modulo* 24 *lies in $R(P)$ is in fact a curvature of a circle in P.*

For over a decade, no one doubted this conjecture, and various weaker results which were proven all increased our certainty that the conjecture was true. Specifically, in joint work with Bourgain [4], we showed that the integers that appear as curvatures in a given packing make up a positive fraction of all integers. In [5], Bourgain-Kontorovich showed that the integers that are admissible according to their residue class modulo 24, but nonetheless do not appear as curvatures in a given packing make up a zero density of all integers: more precisely, that the number of such badly behaved admissible integers $< X$ is bounded above by $X^{1-\varepsilon}$ for some ε. Both of these results relied on an observation of Sarnak's in [28] connecting circles in Apollonian packings with shifted binary quadratic forms which we detail in Section 2. Variations and generalizations of these results, as well as generalizations of the Local to Global Conjecture above, appeared in work of Zhang in [34] and our work with Stange and Zhang in [14].

Then, in the summer of 2023, during an REU exploring some related properties of Apollonian circle packings, Haag, Kertzer, Rickards, and Stange [19] stumbled upon, first by numerical experimentation and then with a proof, the fact that in one of the packings explored in [13], one never gets any integers that are of the form n^2 or $3n^2$, and hence, indeed, infinitely many admissible curvatures do not appear, and the Local to Global Conjecture 1.1 is false. This discovery led them to suspect that, just as in various other examples for which the Hasse principle fails, there is an extra obstruction coming from quadratic reciprocity. Indeed, they were able to prove a likely complete result on how and when these obstructions arise, and also found several instances of obstructions coming from quartic reciprocity. With these new obstructions, they formulated the following new version of the local to global conjecture.

Conjecture 1.2 (Modified Local to Global Conjecture for Apollonian Packings, [19]). *Let P be an integral Apollonian circle packing, let $R(P)$ be the set of residue classes modulo* 24 *of curvatures in P, and let $Q(P)$ denote the set of integers ruled out by the quadratic and quartic obstructions detailed in Theorem 4.2. Then there exists $X_P \in \mathbb{Z}$ such that any integer $x > X_P$ whose residue modulo* 24 *lies in $R(P)$, and which is not in $Q(P)$ is in fact a curvature of a circle in P.*

Note that there still are many packings for which the original Local to Global Conjecture may hold (see Section 4 for details), and we are still far from proving any version of the conjecture for any packing.

In this article, we review how the original conjecture came about, how the quadratic and quartic obstructions were originally missed, and why the newly discovered obstructions of [19] occur.

Acknowledgements: We are grateful to the AMS for giving us the opportunity to tell this story, to James Rickards for valuable feedback on an earlier version of this article, and to Katherine Stange for many helpful conversations on this topic, as well as for providing the data presented in Figure 6.

2. Background

So far, we have only a geometric construction of the circles (and hence of the curvatures which may or may not satisfy a local to global principle). It will be convenient to put this construction into an algebraic context, and we do so in this section. One way to interpret the process of inscribing circles described in Section 1 is that at each step we are given four pairwise tangent circles, and if we fix any three of them, there are (according to Apollonius) two circles tangent to all three: one is the fourth circle in the quadruple, and the other might be a circle we have not yet included in our picture. Indeed, these two circles are related to each other geometrically via circle inversion: one circle is the image of the other upon inversion in the *dual circle* passing through the tangency points of the three circles we fixed (see Figure 3 for an illustration of this).

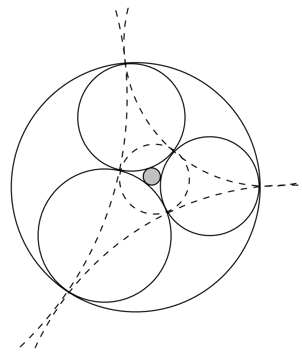


FIGURE 3. In this picture, the shaded circle on the inside is the image of the circle on the outside under inversion in the dual circle passing through the tangency points of the other three circles. Inversion of these circles in the other dual circles would also yield circles in the pictured Apollonian packing.

The curvatures of these two circles have an algebraic relationship as well, which is readily seen in the following theorem that is usually attributed to Descartes, but is more accurately due to both Descartes and Princess Elizabeth of Bohemia, with whom Descartes corresponded about the problem (see [3]).

Theorem 2.1 (Descartes, Princess Elizabeth of Bohemia, 1643)**.** *Let a,b,c,d be curvatures of four pairwise tangent circles, where we take the curvature of a circle internally tangent to the other three to have negative curvature. Then*

$$Q(a,b,c,d) := 2(a^2+b^2+c^2+d^2)-(a+b+c+d)^2 = 0 \tag{2.1}$$

Fixing any three of the variables above, say, a,b,c, one can solve the following quadratic equation for the curvatures of the two possible fourth circles in the tangent quadruple:

$$x^2-2(a+b+c)x+(a^2+b^2+c^2-2ab-2ac-2bc) = 0.$$

We hence see that, if d and d' are the curvatures of the two circles tangent to the circles of curvature a, b, and c, we have

$$d' = 2(a+b+c)-d.$$

In this way, one gets a very nice representation of curvatures in a packing via an orbit of a subgroup of $\mathrm{O}_Q(\mathbb{Z})$, the orthogonal group fixing Q. Performing the above process for the other three possible choices of coordinates to fix, we come up with the following revelation. Given a quadruple of curvatures (a,b,c,d) in an Apollonian packing, the set of all quadruples in the packing is precisely the orbit of the group generated by

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

acting on $(a,b,c,d)^T$ via matrix multiplication on the left. This group, called the *Apollonian group* and first discovered by Hirst [20], is a so-called *thin group* – it is Zariski dense in $\mathrm{O}_Q(\mathbb{C})$ but infinite index in $\mathrm{O}_Q(\mathbb{Z})$ (see [11] for a proof of this), and governs all of the arithmetic and indeed geometry of Apollonian packings. It is now also clear why if one has a circle packing where one quadruple of pairwise tangent circles all have integer curvatures, then every circle in the packing has integer curvature. From now on, we deal only with *primitive* Apollonian packings, i.e. those in which the greatest common divisor of all the curvatures in the packing is 1.

Studying the structure of subgroups of the Apollonian group generated by three of the four generators above, Sarnak made an observation in [28] which is central to nearly all arithmetic results on Apollonian circle packings to date. His result, stated below, associates every circle of curvature a of a given packing with a binary quadratic form of discriminant $-4a^2$.

Proposition 2.2 ([28]). *Let C_a be a circle of curvature a which is a member of a quadruple of pairwise tangent circles with curvatures (a,b,c,d). Let S be the set of circles tangent to C_a. Define*

$$f_{C_a}(x,y) - a = (b+a)x^2 + (a+b+d-c)xy + (d+a)y^2 - a.$$

Then the multiset of curvatures of the circles in S is exactly the multiset of values $f_a(x,y) - a$ where x and y are relatively prime integers.

This result has been used in many subsequent works on Apollonian packings. Of note in the local to global story, the strategy in [4] is to consider all integers appearing as curvatures of circles two tangencies away from a fixed circle – equivalently, integers represented by the corresponding family of shifted binary forms – and to show that these integers make up a positive fraction of $\mathbb{Z}$. In [5], every element of the Apollonian group is associated with such a shifted binary form and the integers occurring as curvatures in the packing are then those integers represented by all the forms corresponding to the Apollonian group. In nearly all subsequent local to global results akin to that in [5], one takes advantage of the presence of this phenomenon: that the integers one studies can be viewed as those represented by an infinite family of shifted binary forms. Ironically, this phenomenon is also a key input into the proof that the local to global conjecture for Apollonian packings is in fact false in [19].

3. The Local to Global Conjecture

With our algebraic way of expressing all quadruples of curvatures of pairwise tangent circles in the packing, we can now ask what quadruples a given packing can contain modulo a prime p. For example, if we look modulo 5, the packings in Figure 2 both have 144 different quadruples of curvatures of pairwise tangent circles; indeed, these are exactly all nontrivial solutions to (2.1) modulo 5. The same can be said modulo 7, 11, 13, and indeed any prime $p \geq 5$. The idea is to consider the Apollonian group – actually, its preimage in the spin double cover of $\mathrm{SO}_{\mathbb{R}}(3,1)$ – modulo various primes p. One has to work in this slightly different setting because A is a subgroup of the orthogonal group $\mathrm{O}_{\mathbb{R}}(3,1)$ where strong approximation does not hold, and it is difficult to say anything about the projection of A into $\mathrm{O}_Q(\mathbb{Z}/p\mathbb{Z})$ by working in the orthogonal group alone. However, the preimage of

A under the spin homomorphism (see [12]) is a Zariski dense subgroup of $\mathrm{SL}_2(\mathbb{C})$ where general results regarding strong approximation are known (see [32], for example). Most relevantly for us, because of the Zariski density of A, one gets that the canonical homomorphism from the preimage of A under the spin homomorphism to $\mathrm{SL}_2(\mathbb{Z}(\sqrt{-1})/\mathfrak{p})$ is surjective for all but finitely many primes $\mathfrak{p}$. With some work, we determine that these finitely many primes are only $\mathfrak{p}^2 = (2)$ and $\mathfrak{p} = (3)$, and, moreover, one can use a combination of Goursat's Lemma and versions of Hensel's Lemma to completely describe the group's behavior modulo d for any $d > 1$. In a nutshell, everything is as multiplicative as one could hope for, and the serious obstruction is modulo 8 and 3 (once one understands the situation modulo 8, one lifts naturally to higher powers of 2, and similarly for 3. Specifically, we proved the following.

Theorem 3.1 ([12])**.** *Let $\mathscr{P}$ be an orbit of A acting on the root quadruple $\mathbf{v}_P$ of a primitive packing P and let $\mathscr{P}_d$ be the reduction of this orbit modulo an integer $d > 1$. Let*

$$C_d = \{\mathbf{v} \in \mathbb{Z}/d\mathbb{Z} \,|\, \mathbf{v} \not\equiv \mathbf{0}\,(d),\, Q(\mathbf{v}) \equiv 0\,(d)\}.$$

Write $d = d_1 d_2$ with $(d_2, 6) = 1$ and $d_1 = 2^n 3^m$ where $n, m \geq 0$.

(i) *The canonical projection $\mathscr{P}_d \longrightarrow \mathscr{P}_{d_1} \times \mathscr{P}_{d_2}$ is surjective.*
(ii) *The canonical projection $\mathscr{P}_{d_2} \longrightarrow \prod_{p^r || d_2} \mathscr{P}_{p^r}$ is surjective and $\mathscr{P}_{p^r} = C_{p^r}$.*
(iii) *The canonical projection $\mathscr{P}_{d_1} \longrightarrow \mathscr{P}_{2^n} \times \mathscr{P}_{3^m}$ is surjective.*
(iv) *If $n \geq 4$, let $\pi : C_{2^n} \to C_8$ be the canonical projection. Then $\mathscr{P}_{2^n} = \pi^{-1}(\mathscr{P}_8)$.*
(v) *If $m \geq 2$, let $\phi : C_{3^m} \to C_3$ be the canonical projection. Then $\mathscr{P}_{3^m} = \phi^{-1}(\mathscr{P}_3)$.*

In other words, if one computes an Apollonian orbit modulo 24, one can easily infer what the orbit looks like modulo any integer using this information coupled with solutions to (2.1) modulo prime powers. Given this theorem, it is natural to ask whether outside of this obstruction modulo 24, no other issues arise. For example, the first packing in Figure 2 only has integers that are in the set $\{2, 3, 6, 11, 14, 15, 18, 23\}$ modulo 24. Could it be that any large enough integer in this admissible set is in fact a curvature in this packing? In [13], we set out to see whether numerical evidence would support Conjecture 1.1, which predicts a local to global principle for the curvatures in a given integral Apollonian packing.

Given computational limitations and our assumption that all integral packings probably exhibit similar behavior as far as the local to global principle is concerned, we considered two packings: P_B generated by the quadruple $(-1, 2, 2, 3)$, and P_C, generated by the quadruple $(-11, 21, 24, 28)$ which are shown in Figure 2, in that order.

In order to explain the data we obtain in both cases, and to shed more light upon the local properties of curvatures in an Apollonian packing which one *can* prove, we first study the packings P_B and P_C modulo 24. Recall that the Apollonian group A is generated by the four generators S_i described in Section 2. We can view an orbit of A modulo 24 as a finite graph $\mathscr{G}_{24}$ in which each vertex corresponds to a distinct (mod 24) quadruple of curvatures, and two vertices $\mathbf{v}$ and $\mathbf{v}'$ are joined by an edge if and only if $S_i \mathbf{v} = \mathbf{v}'$ for some $1 \leq i \leq 4$. According to Theorem 3.1, given any orbit $\mathscr{P}$ of the Apollonian group, we have

$$\mathscr{P}_{24} = \mathscr{P}_8 \times \mathscr{P}_3,$$

so the graph $\mathscr{G}_{24}$ is completely determined by the structure of $\mathscr{P}_3$ and $\mathscr{P}_8$. There are only two possible orbits modulo 3, pictured in Figure 4, the first of which corresponds to P_B, and the second to P_C. There are many possible orbits modulo 8, and we provide the graphs for these orbits in the case of P_B and P_C in Figure 5.

Given the structure of these graphs, one sees that the curvatures of circles in a packing modulo 24 are equally distributed among the coordinates of the vertices in $\mathscr{G}_{24}$. Combined

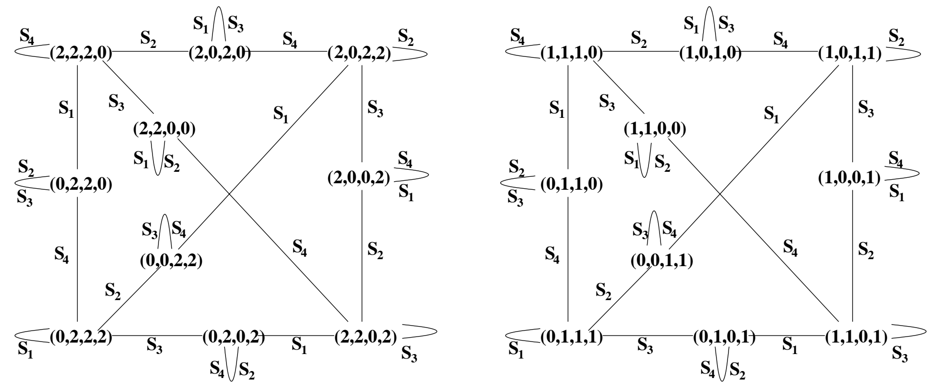


FIGURE 4. Orbits of P_B and P_C modulo 3

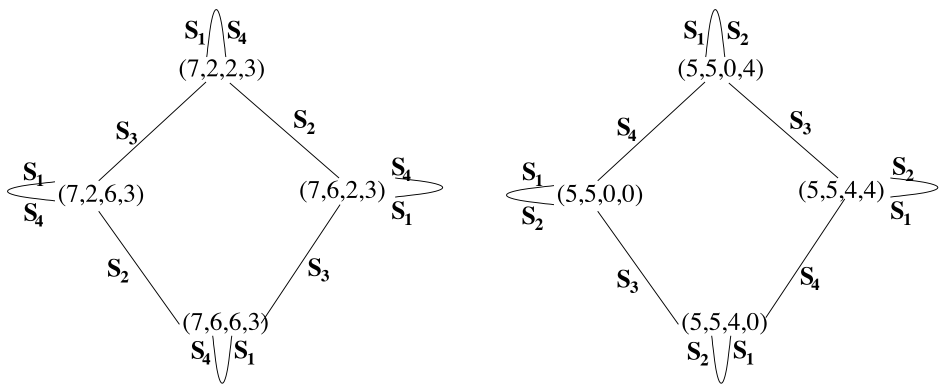


FIGURE 5. Orbits of P_B and P_C modulo 8

with Theorem 3.1, this lets us compute the ratio of curvatures in a packing which fall into a specific congruence class modulo 24. Namely, let $\mathscr{P}_{24}(P)$ be the orbit mod 24 corresponding to a given packing P. For $\mathbf{w} \in \mathscr{P}_{24}(P)$ let w_i be the ith coordinate of $\mathbf{w}$. We then define

$$\gamma(n,P) = \frac{\sum_{i=1}^{4} \#\{w \in \mathscr{P}_{24}(P) |\ w_i = n\}}{4 \cdot \#\{\mathbf{w} \in \mathscr{P}_{24}(P)\}}. \tag{3.1}$$

With this notation, a packing P contains a circle of curvature congruent to n modulo 24 if and only if $\gamma(n,P) > 0$. We express γ as follows:

$$\begin{aligned} \gamma(n,P) &= \frac{\sum_{i=1}^{4} \#\{\mathbf{w} \in \mathscr{P}_{24}(P) |\ w_i = n\}}{4 \cdot \#\{\mathbf{w} \in \mathscr{P}_{24}\}} \\ &= \frac{\sum_{i=1}^{4} \#\{\mathbf{w} \in \mathscr{P}_{8} |\ w_i \equiv n\ (3)\} \cdot \#\{w \in \mathscr{P}_{3} |\ w_i \equiv n\ (8)\}}{4 \cdot \#\{\mathbf{w} \in \mathscr{P}_{8}\} \cdot \#\{\mathbf{w} \in \mathscr{P}_{3}\}}. \end{aligned} \tag{3.2}$$

Theorem 3.1 then implies that, if $N_P(X)$ is the total number of circles of curvature $< X$ in the packing P, then

$$\sum_{\substack{C\in P \\ a(C)<x \\ a(C)\equiv n\,(24)}} 1 \sim \gamma(n,P)\cdot N_P(x)$$

where C denotes a circle, and $a(C)$ its curvature. Note that Kontorovich-Oh showed in [22] that

$$N_P(X) \sim c_P X^{\delta}$$

where P is any packing – not necessarily integral – the constant c_P depends on P, and $\delta = 1.3056\ldots$ is the Hausdorff dimension of P (this value is the same regardless of the packing), answering a question of Boyd in [6]. In general, the orbits $\mathscr{P}_8(P)$ and $\mathscr{P}_3(P)$ have 4 and, respectively, 10 vertices in the corresponding finite graphs. Therefore $\mathscr{G}_{24}$ always has 40 vertices, and the ratio in (3.2) is easily computed using this graph. Indeed, we have the following facts about the packings P_B nd P_C. Letting $P_B(24)$ and $P_C(24)$ denote the residue classes mod 24 in the packing P_B and, respectively, P_C, we have

- $P_B(24) = \{2,3,6,11,14,15,18,23\}$
- $P_C(24) = \{0,4,12,13,16,21\}$
- $\gamma(2,P_B) = \gamma(11,P_B) = \gamma(14,P_B) = \gamma(23,P_B) = \frac{3}{20}$, $\gamma(3,P_B) = \gamma(6,P_B) = \gamma(15,P_B) = \gamma(18,P_B) = \frac{1}{10}$
- $\gamma(13,P_C) = \frac{3}{10}$, $\gamma(21,P_C) = \frac{1}{5}$, $\gamma(4,P_C) = \gamma(16,P_C) = \frac{3}{20}$, $\gamma(0,P_C) = \gamma(12,P_C) = \frac{1}{10}$

(i) $N_{P_B}(x) \sim c_{P_B}\cdot x^{\delta}$, where $c_{P_B} = 0.402\ldots$

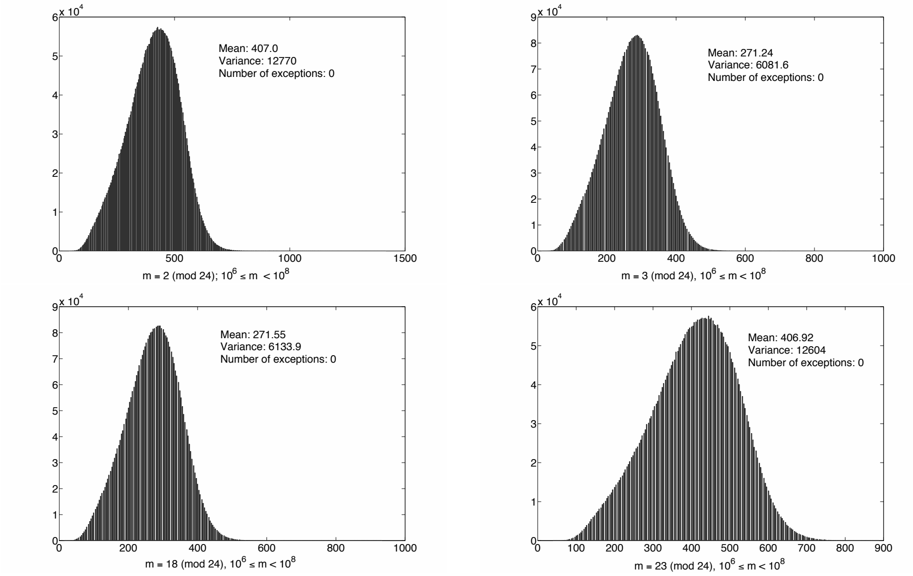
(i) $N_{P_C}(x) \sim c_{P_C}\cdot x^{\delta}$, where $c_{P_C} = 0.0176\ldots$

(iv) For $10^6 < x < 5\cdot 10^8$, let x_{24} denote x mod 24. If $x_{24} \in P_{B,24}$ then x is a curvature in the packing P_B.

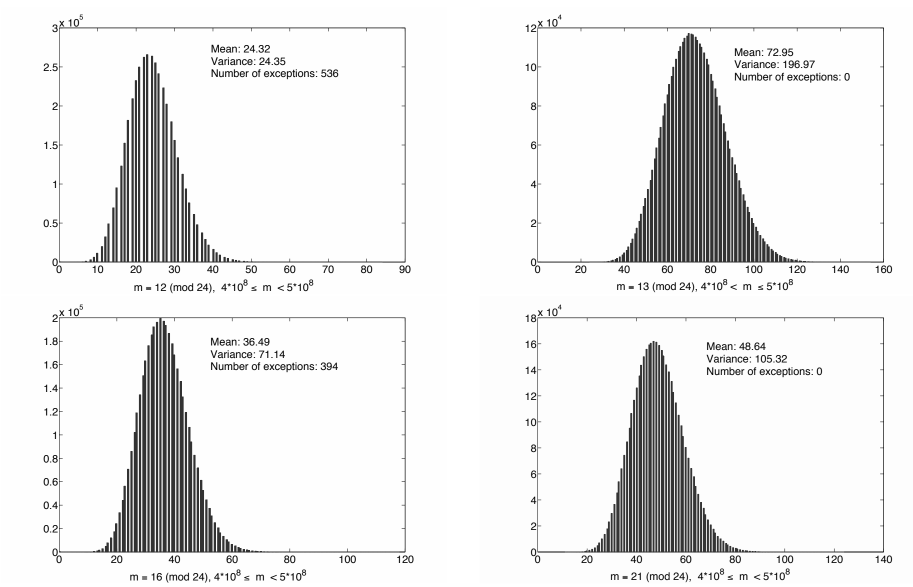
(iv) For $10^8 < x < 5\cdot 10^8$, let x_{24} denote x mod 24. If $x_{24} = 13$ or $x_{24} = 21$, then x is a curvature in the packing P_C.

Note that $\gamma(n,P_B) = \gamma(n+12,P_B)$ – for this particular packing, one can hence express the local obstructions modulo 12 rather than modulo 24, an observation made by Graham-Lagarias-Mallows-Wilks-Yan in [18]. Whenever this is the case for an integral Apollonian packing, we will find that there are eight congruence classes modulo 24 in the curvatures of the circles.

We then generated a number of histograms as follows. For various large intervals $[X,Y]$, given an admissible reside class r modulo 24 in a packing P, we considered the set $S_r(X,Y)$ of all integers in $[X,Y]$ congruent to r modulo 24 and, upon computing all curvatures in $[X,Y]$ (with multiplicity) in a packing P, we recorded how many integers in $S_r(X,Y)$ appeared n times as a curvature in P, for $n \geq 0$. We recorded this in a histogram where the x-axis represents frequencies n, and the bars represent the number of integers in $S_r(X,Y)$ corresponding to a given range of frequencies. In general, we saw that, for X relatively small there was always a bar above $n = 0$ (meaning that there were some integers in $S_r(X,Y)$ not appearing as a curvature in P), as X and Y grew that bar shrank and the mean of the frequencies shifted to the right. In the case of P_B, we found that given $[X,Y] = [10^6,10^8]$, there was no longer a bar above $n = 0$ at all for any of the admissible congruence classes m, meaning that all integers in $S_r(X,Y)$ appeared as curvatures in P_B, as is apparent in some of the histograms for P_B which follow.



In the case of P_C, we considered the interval $[X,Y] = [4 \cdot 10^8, 5 \cdot 10^8]$ (which is as far out as our program allowed us to go), and found that there was still a small bar over $n = 0$ for four of the six residue classes, and no bar over $n = 0$ for $r = 13$ and 21 modulo 24, meaning that all integers in $S_{13}(X,Y)$ and $S_{21}(X,Y)$ appeared as curvatures in P_C, but not so for the other values of r. We show some of these histograms for P_C below.



Note that there are several frequencies not represented in the histograms for both P_B and P_C (these show up as gaps in the graphs), some of which are due to formatting of the histograms, and some coming from symmetry in the case of P_B, but we will focus on frequency 0 here. At the time we ran these experiments, we assumed that the fact that we were still seeing a bar over $n = 0$ for some residue classes in the P_C histograms was due to the fact that growth in P_C is a bit faster than in P_B: specifically,

$$N_{P_B}(x) \sim c_{P_B} \cdot x^{\delta}, \text{ where } c_{P_B} = 0.402\ldots, \text{ and } N_{P_C}(x) \sim c_{P_C} \cdot x^{\delta}, \text{ where } c_{P_C} = 0.0176\ldots,$$

which might be noticeable on the level that we were examining the packings. Especially given the encouraging data for the packing P_B, we surmised that this could mean that one needs to consider X and Y larger than our program could handle in order to see the histograms shift away from 0 completely to the right in the case of P_C. In retrospect, the fact the histograms for the problematic residue classes for P_C had a persistent bar over $n = 0$, even while other small frequencies were no longer occurring, should have been a red flag for anyone looking at these histograms. Interestingly, this is not what led anyone to suspect that local to global might fail in some cases.

4. Quadratic and quartic obstructions

In the summer of 2023, Stange and Rickards ran an REU at the University of Colorado, Boulder. The question they were studying was,

For which pairs of curvatures does there exist a packing containing both at once?

The students involved with the project – Summer Haag and Clyde Kertzer – put together some data on this question as follows. Let A and B be two congruence classes modulo 24 which are not ruled out as occurring in some packing together. Let curvatures $24x+A$ from class A run along the x-axis, and curvatures $24y+B$ from class B run along the y-axis, and plot a point (x,y) whenever there is no packing that has both curvatures $24x+A$ and $24y+B$ together. According to the local to global conjecture, the plotted points should peter out as one goes further out, but the students' data contradicted this. Indeed, they found one graph in which the points persisted and seemed to follow a quadratic pattern. See Figure 6, which shows these points persisting in a pattern as one goes out further in the case of A being the class 0 modulo 24, and the class B being 8 modulo 24. Notably, out of all possible pairs (A,B), the pair (0.8) was the only one exhibiting this behavior, and if they had only checked some of the graphs, and neglected this one, this is where our story would end.

Thinking at first that this must be a bug in the program, Haag-Kertzer-Rickards-Stange soon realized it was actually a missed obstruction which was not a congruence obstruction, and did not come from the Zariski closure of the group, which is in a sense what governs the obstructions modulo 24. Indeed, in our work with Sanden we never thought to collect the *missing* curvatures in either of the packings we considered. It simply did not occur to us that the missing curvatures may be following some pattern, especially given the positive data for P_B. If we had, we would have quickly realized that the packing P_C never has any curvatures of the form n^2 or $3n^2$ in it, and so there are infinitely many curvatures that are $0,4,12,$ and 16 modulo 24 that are missed in that packing, despite those residues being admissible. Indeed, these account for all the missed curvatures contributing to the bar above 0 in the histograms above.

We now give an overview of how these quadratic (and quartic) obstructions occur. It is difficult to pinpoint exactly what about the Apollonian group gives rise to these obstructions, but they come both from properties of the group that one would consider rich, and from the particular way in which residues modulo 24 are grouped for various packings. More on that later.

[2] This plot, generated by Summer Haag, was shared with the author by Katherine Stange

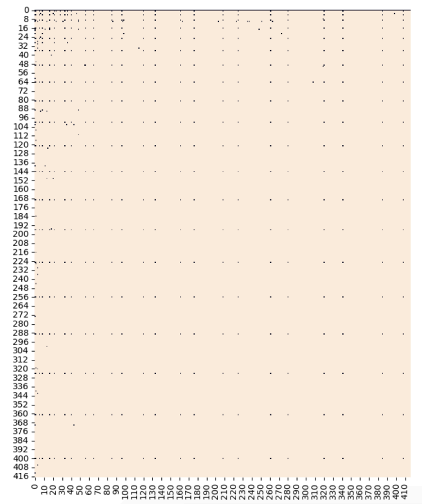


FIGURE 6. Missing pairs of curvatures that are 0 and 8 modulo 24, respectively, across all packings. [2]

The first major observation that is made in [19] is a more precise version of what was known (but not published) before – that every primitive integral packing admits either six or eight possible congruence classes modulo 24. Specifically, they show the following.

Proposition 4.1 ([19]). *Let $R(P)$ be the set of residues modulo 24 of the curvatures in a packing P. Then $R(P)$ is one of six possible sets, labeled by type as follows:*

Type	$R(P)$
$(6,1)$	$0,1,4,9,12,16$
$(6,5)$	$0,5,8,12,20,21$
$(6,13)$	$0,4,12,13,16,21$
$(6,17)$	$0,8,9,12,17,20$
$(8,7)$	$3,6,7,10,15,18,19,22$
$(8,11)$	$2,3,6,11,14,15,18,23$

The set $R(P)$ is the admissible set *for the packing. The type (x,k) denotes that $R(P)$ has cardinality x and k is the smallest positive residue in $R(P)$ coprime to* 24.

It is significant to note that for all the types where $x=6$ there is precisely one residue class that is coprime to 24, and for the cases where $x=8$ there are two such residue classes.

The authors of [19] then show that there is an invariant of any given packing P which they denote by $\chi_2(P)$ which takes on the values ± 1 and determines the quadratic obstructions for the packing. For some packing types, they also define an invariant $\chi_4(P)$ governing quartic obstructions, which takes on values $1, i, -1$, and i, such that $\chi_4(P)^2 = \chi_2(P)$. A crucial part of the proof of their result is to show that the functions χ_2 and χ_4, which they define on circles of a packing P, are in fact constant over the whole packing.

Given this, they refine the types (x,k) of packings in Proposition 4.1 to types (x,k,χ_2) or, when the given packing also comes with a quartic obstruction, (x,k,χ_2,χ_4), and state their main theorem as follows.

Theorem 4.2 ([19]). *The table below describes quadratic and quartic obstructions that exist for the type (x,k,χ_2,χ_4) (or, in some cases, (x,k,χ_2)). Specifically, for a packing of a given type in one of the rows below, the second and third column indicate integers which do not appear as curvatures in a packing of the corresponding type. The last two columns indicate residue classes modulo 24 for which the local to global conjecture fails, or may yet hold.*[3]

Type	*Quadratic Obstructions*	*Quartic Obstructions*	*LG fails in classes*	*LG may hold in classes*
$(6,1,1,1)$	*none*	*none*	*none*	$0,1,4,9,12,16$
$(6,1,1,-1)$	*none*	$n^4, 4n^4, 9n^4, 36n^4$	$0,1,4,9,12,16$	*none*
$(6,1,-1)$	$n^2, 2n^2, 3n^2, 6n^2$	*none*	$0,1,4,9,12,16$	*none*
$(6,5,1)$	$2n^2, 3n^2$	*none*	$0,8,12$	$5,20,21$
$(6,5,-1)$	$n^2, 6n^2$	*none*	$0,12$	$5,8,20,21$
$(6,13,1)$	$2n^2, 6n^2$	*none*	0	$4,12,13,16,21$
$(6,13,-1)$	$n^2, 3n^2$	*none*	$0,4,12,16$	$13,21$
$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0,9,12$	$8,17,20$
$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0,9,12$	$8,17,20$
$(6,17,-1)$	$n^2, 2n^2$	*none*	$0,8,9,12$	$17,20$
$(8,7,1)$	$3n^2, 6n^2$	*none*	$3,6$	$7,10,15,18,19,22$
$(8,7,-1)$	$2n^2$	*none*	18	$3,6,7,10,15,19,22$
$(8,11,1)$	*none*	*none*	*none*	$2,3,6,11,14,15,18,23$
$8,11,-1)$	$2n^2, 3n^2, 6n^2$	*none*	$2,3,6,18$	$11,14,15,23$

[3]other quadratic and quartic obstructions are not ruled out by this theorem, but data collected by the authors of [19] gives a strong indication that these are the only ones.

Note that there are two packing types for which the original Local to Global Conjecture as stated in Conjecture 1.1 may still hold – types $(6,1,1,1)$ and $(8,11,1)$. The packing P_B is of type $(6,1,1,1)$, and the packing P_C is of type $(6,13,-1)$, where we see that the Local to Global Conjecture may still hold for residue classes 13 and 21, as our data in [13] indicates.

The proof of this theorem relies on Proposition 2.2 and the classification of congruence classes mod 24 in Apollonian packings described in Table 4.1. Because the proofs of the existence of the quartic obstructions and the existence of the quadratic obstructions in the table above are very similar, we give an overview of how the quadratic obstructions arise and refer the reader to [19] for details on the quartic ones.

Before we delve into the details, we recall some facts about the Kronecker symbol $\left(\frac{\cdot}{\cdot}\right)$ and quadratic reciprocity in this setting.

Definition 4.3. Let m,n be positive integers with $n>1$ for the purpose of this article, and let $n = 2^{e_0}p_1^{e_1}\cdots p_k^{e_k}$ be the prime factorization of n. Then the *Kronecker symbol* $\left(\frac{m}{n}\right)$ is defined to be

$$\left(\frac{m}{n}\right) = \prod_{i=0}^{k}\left(\frac{m}{p_i}\right)^{e_i},$$

where

$$\left(\frac{m}{p}\right) := \begin{cases} 0 & \text{if } p|m \\ 1 & \text{if } p \text{ is odd and } m \text{ is a quadratic residue (mod } p) \\ 1 & \text{if } p=2 \text{ and } m \equiv \pm 1 \pmod 8 \\ -1 & \text{otherwise} \end{cases}$$

With this definition, we have the following, which is known as *quadratic reciprocity* for the Kronecker Symbol. Let $m = 2^{e_0}m'$ and $n = 2^{f_0}n'$ where m',n' are odd. Then

$$\left(\frac{m}{n}\right) = (-1)^{\frac{m'-1}{2}\frac{n'-1}{2}}\left(\frac{n}{m}\right) \tag{4.1}$$

We are now ready to return to the question of quadratic obstructions in Apollonian packings. First, given a primitive binary quadratic form $f(x,y)$ of discriminant $-4a^2$ as in Proposition 2.2, Haag-Kertzer-Rickards-Stange show

Lemma 4.4 ([19])**.** *There exists a unique primitively represented and invertible residue $f(x,y)$ modulo a, up to multiplication by a square. Denote any lift of this value to the positive integers by $\rho(f)$.*

Now, given a circle C of curvature a in a packing P, let $\rho = \rho(f_C)$ where f_C is as in Proposition 2.2 and define

$$\chi_2(C) = \begin{cases} \left(\frac{\rho}{a}\right) & \text{if } a \equiv 0,1 \pmod 4 \\ \left(\frac{-\rho}{a/2}\right) & \text{if } a \equiv 2 \pmod 4 \\ \left(\frac{2\rho}{a}\right) & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

Note that the choice of $\rho(f_C)$ does not affect the value of χ_2 since $\rho(f_C)$ is uniquely defined up to multiplication by a square and addition of multiples of a. With this definition, the first key ingredient in the proof of the quadratic obstructions in Theorem 4.2 is the following result.

Proposition 4.5 ([19])**.** *Let C_1, C_2 be tangent circles in a packing P with coprime curvatures a and b, respectively. Then $\chi_2(C_1) = \chi_2(C_2)$.*

For a sense of why this is true, note that the fact that C_1 and C_2 are tangent exactly means, by Proposition 2.2, that $a+b$ is represented primitively both by f_{C_1} and f_{C_2}, and hence we can take $\rho(f_{C_1}) = a+b = \rho(f_{C_2})$. If P is of type $(6,k)$ for some k, then all curvatures in P are 0 or 1 mod 4, and in particular a,b are either 0 or 1 mod 4 and not both even. Quadratic reciprocity in (4.1) then implies that

$$\chi_2(C_1)\chi_2(C_2) = \left(\frac{a+b}{a}\right)\left(\frac{a+b}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1, \tag{4.2}$$

and, since $\chi_2(C_1)$ and $\chi_2(C_2)$ are both 1 or -1, they must be equal. Note that here it is crucial that we happen to have only curvatures that are 0 or 1 (mod 4) in the packing. If somehow $0,1,2$, and 3 mod 4 were mixed into packings in some other way, this would fall apart, and we might get -1 instead of 1 in (4.2). One handles packing of type $(8,k)$ similarly (note that in such packings only residues 2 and 3 (mod 4) arise).

In fact, one has more: indeed, that χ_2 must be constant for the whole packing P.

Proposition 4.6 ([19])**.** *The value of χ_2 is constant across all circles in a fixed primitive Apollonian packing P. Denote this value by $\chi_2(P)$.*

This proposition follows from Proposition 4.5 and the fact that for any pair of circles C, C' in P there exists a path of tangent circles from C to C' where consecutive circles in the path have coprime curvatures – another property of Apollonian packings that is readily seen by characterizing certain circles in the packing via a quadratic polynomial: this time, the relevant fact is that, given tangent circles C_1 and C_2 of curvature a and b, respectively, the curvatures of the family of circles tangent to both C_1 and C_2 are the values of the quadratic polynomial

$$f(x) = (a+b)x^2 - (a+b+c-d)x + c$$

with $x \in \mathbb{Z}$, and c,d correspond to curvatures of some fixed two circles C_3 and C_4, respectively, so that C_1, C_2, C_3, and C_4 are pairwise tangent to one another. This appears in [19] as Lemma 4.5, and also appears in slightly different forms in [18], [11], and most recently in [10]. It is not hard to show that the polynomial above represents an integer that is coprime to ab, or, equivalently, that there is a circle tangent to C_1 and C_2 whose curvature is coprime to a and b. Furthermore, one can show that given any circle C of curvature a in a primitive packing, it is tangent to a circle C_2 of curvature b where $6|ab$. Combining this with the above, one has the following lemma.

Lemma 4.7 ([19])**.** *Given a circle C of curvature a in a primitive packing P, there is a circle C' tangent to it which has curvature coprime to $6a$.*

Now that we have collected these facts, we are ready to show the existence of the quadratic obstructions listed in Theorem 4.2. Essentially, the congruence classes modulo 24 that are allowed to occur together in one packing, combined with the fact that χ_2 is constant over the whole packing force the impossibility of some integers in that packing. The flavor is very similar to an example of Iskovskikh [21], who showed that

$$y^2 + z^2 = (3-x^2)(x^2-2)$$

fails the local to global principle in that it has local solutions everywhere, but not global ones (over $\mathbb{Q}$). This obstruction is an example of a broader type called a Brauer-Manin obstruction, which we will not get into here.

Coming back to Apollonian packings, we write out the argument for packing types $(6,k)$ here. The obstructions for types $(8,k)$ occur in a very similar way. Suppose P is of type

$(6,k)$, and there is a circle of curvature un^2 in P. By Lemma 4.7, we have that this circle is tangent to a circle C of curvature m which is coprime to $6un^2$. Given that k is the only congruence class modulo 24 which is coprime to 6 in a packing of type $(6,k)$, we have that $m \equiv k \pmod{24}$. Furthermore, by Proposition 4.6, the definition of χ_2, and properties of the Kronecker symbol, we have

$$\left(\frac{un^2}{m}\right) = \left(\frac{u}{m}\right) = \left(\frac{\rho(f_C)}{m}\right) = \chi_2(P).$$

Now, suppose $k = 1$, i.e. $m \equiv 1 \pmod 8$ and $\pmod 3$. By quadratic reciprocity in (4.1), one has

$$\left(\frac{2}{m}\right) = \left(\frac{m}{2}\right),$$

and since $m \equiv 1 \pmod 8$ one has $\left(\frac{m}{2}\right) = 1$. Similarly, using quadratic reciprocity and $m \equiv 1 \pmod 3$, one obtains

$$\left(\frac{3}{m}\right) = \left(\frac{m}{3}\right) = 1,$$

and so numbers of the form $2n^2, 3n^2$, and $6n^2$ can exist in packings where $\chi_2(P) = 1$, but not where $\chi_2(P) = -1$, as is seen in the third row of the table in Theorem 4.2. Using this quadratic reciprocity argument combined with m's residue modulo 8 and 3, one determines that, for $k = 5$ one has $\left(\frac{2}{m}\right) = -1, \left(\frac{3}{m}\right) = -1$, and hence $\left(\frac{6}{m}\right) = 1$. Thus, if $\chi_2(P) = 1$, curvatures of the form $6n^2$ can appear but curvatures of the form $2n^2$ and $3n^2$ cannot, while if $\chi_2(P) = -1$ one can have curvatures of the form $2n^2$ and $3n^2$, but not $6n^2$. For $k = 13$ one has $\left(\frac{2}{m}\right) = -1, \left(\frac{3}{m}\right) = 1$, and hence $\left(\frac{6}{m}\right) = -1$. For k=17, one has $\left(\frac{2}{m}\right) = 1, \left(\frac{3}{m}\right) = -1$, and hence $\left(\frac{6}{m}\right) = -1$. Furthermore, whenever $\chi_2(P) = -1$, one cannot have any curvatures of the form n^2. This gives all the quadratic obstructions described in Theorem 4.2 for packings of type $(6,k)$.

Note that the quadratic obstructions arise in harmony with the local obstructions: outside the direct obstruction which does not allow squares to appear, the quadratic obstructions show up specifically for numbers of the form un^2 where $u = 2, 3$, or 6 (and, indeed, the quartic obstructions are also combined with numbers whose prime factorizations involve only 2 and 3). It is tantalizing that the residues modulo 24 which are present together in any one type of packing are almost on purpose forcing these quadratic obstructions, and it would be particularly satisfying to understand exactly what about the Apollonian group makes this happen.

Rickards and Stange have also recently proven the occurrence of such quadratic obstructions in certain semigroup orbits in $\mathrm{SL}_2(\mathbb{Z})$ in [27], with implications in problems related to Zaremba's conjecture for continued fractions. But this is a subject for a different paper.

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