A NOTE ON THE DENSITY OF CURVATURES IN INTEGER APOLLONIAN CIRCLE PACKINGS

ELENA FUCHS

ABSTRACT. Bounded Apollonian circle packings (ACP's) are constructed by repeatedly inscribing circles into the triangular interstices in a configuration of four mutually tangent circles, one of which is internally tangent to the other three. An integer ACP is one in which all of the circles have integer curvature. In this paper, we give a lower bound on $\kappa(P,X)$, the number of integers less than X occurring as curvatures in a bounded integer ACP P.

1. INTRODUCTION

Starting with three mutually tangent circles inscribed in a large circle as in the first picture in Figure 1, one can inscribe a smaller circle into each of the curvilinear triangles between the circles as in the second picture. By an old theorem of Apollonius of Perga (circa 200 BC), there is a unique way of doing this – he discovered that to any three mutually tangent circles there are precisely two other circles that are tangent to all three. We can continue packing circles in this way (see the third picture in Figure refempty), creating smaller curvilinear triangles each time. This process continues indefinitely, and we thus get an infinite packing of circles known as the Apollonian circle packing (ACP). Since the radii of the circles in an ACP get extremely small after several iterations, it is convenient to consider the circles' curvatures, the reciprocals of the radii, instead.

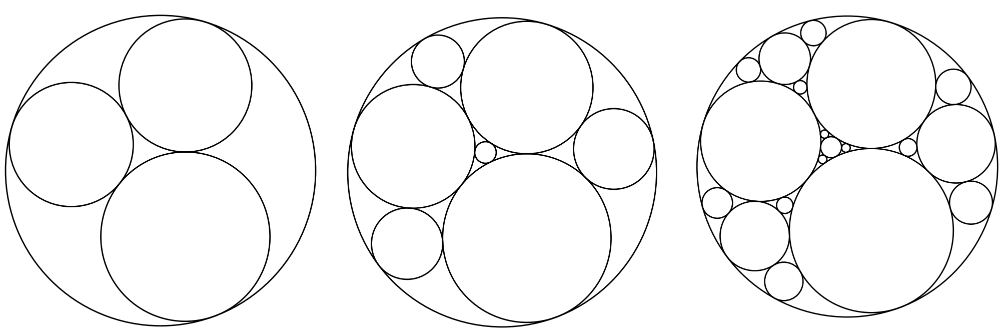

FIGURE 1. Apollonian Circle Packings

A remarkable feature of these packings is that, given a packing in which any four mutually tangent circles have integer curvature, all of the circles in the packing will have integer curvature. Such a packing is called an integer ACP and is illustrated in Figure 2 with the packing generated by starting with circles of curvatures $1,2,2$, and 3. In their paper [GLMWY], the five authors Graham, Lagarias, Mallows, Wilks, and Yan address various natural number-theoretic questions associated with integer ACP's. They make considerable progress in treating the problem, and ask several fundamental questions, many of which have been resolved recently in [F], [FS], [BF], and [KO].

One question Graham at al. ask in [GLMWY] is whether the integers represented as curvatures in a given packing P make up a positive fraction of $\mathbb{N}$. They prove that the number $\kappa(P,X)$, the number of such integers less than X, is bounded below by $\sqrt{X}$:

$$\kappa(P,X) > c\sqrt{X} \tag{1.1}$$

where $c > 0$ is a constant depending on P. They remark, however, that this bound is fairly loose – in fact, they conjecture that $\lim_{X\to\infty} \frac{\kappa(P,X)}{X} > 0$ for any integer packing P. Sarnak improves their bound in [S2] by showing

$$\kappa(P,X) > \frac{cX}{\sqrt{\log X}} \tag{1.2}$$

where $c > 0$ depends on P. In this paper, we use Sarnak's approach in [S2] to refine the bound in (1.2) in the following theorem.

Theorem 1.1. *For a bounded integer Apollonian circle packing P, let $\kappa(P,X)$ denote the number of distinct integers up to X occurring as curvatures in the packing. Then we have*

$$\kappa(P,X) > \frac{cX}{(\log X)^{\varepsilon}},$$

where $\varepsilon = 0.153\ldots$ and $c > 0$ depends on P.

This is not a positive fraction of all integers as conjectured in [GLMWY], but it is a promising result in this direction and can likely be improved with a fine-tuning of the tools we use[1].

It is important to note that this question is different from one recently addressed in [KO] by Kontorovich and Oh about the number $N_P(x)$ of circles in a given packing P of curvature less than X. This involves counting curvatures appearing in a packing with multiplicity, rather than counting every integer which comes up exactly once. In fact, the results in [KO] suggest that the integers occurring as curvatures in a given ACP arise with significant multiplicity. Specifically, they

[1]In a recent paper by the author and Bourgain [BF], the ideas in this paper are in fact fine-tuned to produce a positive fraction of $\mathbb{N}$.

prove $N_P(X)$ is asymptotic to $c \cdot X^{\delta}$, where $\delta = 1.30\ldots$ where c depends on P. Kontorovich and

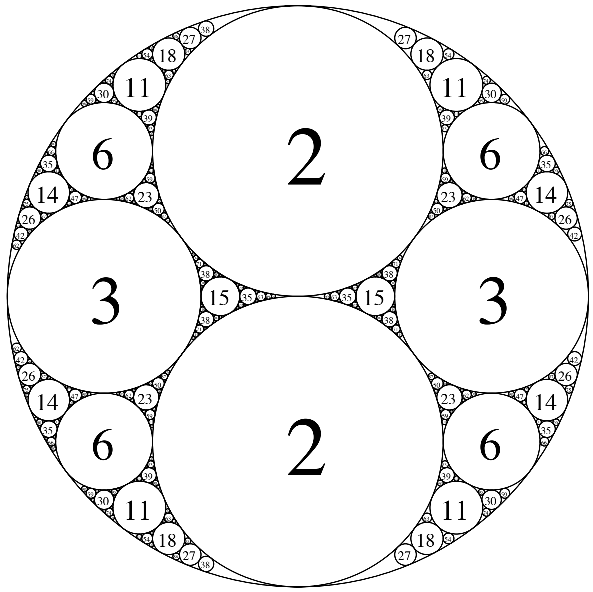

FIGURE 2. Apollonian Circle Packing $(-1,2,2,3)$

Oh's techniques, however, do not extend in any obvious way to counting integers which arise as curvatures in a packing without multiplicity.

Our approach for this is to count curvatures in different "subpackings" of an ACP. Namely, we follow Sarnak's method of fixing a circle C_0 in [S2] to determine a set $\mathscr{A}_0$ of integers which occur as curvatures of circles tangent to C_0 in Section 2. Each integer $a \in \mathscr{A}_0$ then corresponds to a circle C tangent to C_0. In Section 3 we repeat Sarnak's method by fixing circles C of curvature $a \in \mathscr{A}_0$ and counting circles tangent to C. The difficulty with this is that we count many circles more than once in this way, and current upper bounds on how many circles are counted twice are too crude to allow us to do this indefinitely.

Acknowledgements: Many thanks to Peter Sarnak, Jean Bourgain, and Jeff Lagarias for introducing this problem to me, and for numerous insightful comments and conversations.

1.1. **The Apollonian group.** In 1643, Descartes discovered that the curvatures of any four externally cotangent circles of curvatures x_1, x_2, x_3, and x_4 satisfy the equation

$$Q(x_1,x_2,x_3,x_4) = 2(x_1^2+x_2^2+x_3^2+x_4^2) - (x_1+x_2+x_3+x_4)^2. \tag{1.3}$$

This also holds for four mutually tangent circles when three of the circles are internally tangent (or inscribed) to the fourth circle as in the case of ACP's (see [Cx] for a proof). However, the curvature of the outside circle must have a negative sign to satisfy the equation.

Note that Descartes' theorem allows us to solve a quadratic equation for the curvatures of the two circles tangent to three fixed mutually tangent circles. In this way, the formula in (1.3) in fact generates all of the curvatures in a packing P. Specifically, if we assign to every set of 4 mutually tangent circles in a packing P a vector $\mathbf{v} \in \mathbb{Z}^4$ of the circles' curvatures, the process of solving Descartes' equation for the fourth coordinate of $\mathbf{v}$ when three are known gives rise to a matrix group action which generates all quadruples of mutually tangent circles in P. This group, called the Apollonian group and denoted by A, was first discovered by Hirst in [H] is generated by the four matrices

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.4}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix},$$

where $S_i^2 = I$ for $1 \le i \le 4$ and left multiplication of S_i by a vector $\mathbf{v} = (v_1, v_2, v_3, v_4)^T \in \mathbb{Z}^4$ fixes three of the coordinates of $\mathbf{v}$. In [GLMWY], Graham et. al. describe an algorithm for generating a *root quadruple* $\mathbf{v}$ of a packing P, which corresponds to the four largest circles in the packing and is uniquely determined for any given ACP. From this point on, we assume the vector $\mathbf{v}$ is the root quadruple of the packing in question, and that the packing P is primitive – the gcd of the coordinates of $\mathbf{v}$ is 1.

With this notation, the orbit $\mathscr{O} = A\mathbf{v}^T$ represents all of the quadruples of mutually tangent circles[2] in a packing P – the circles' curvatures correspond to the maximum norm of the vectors in $\mathscr{O}$.

Since the quadratic form in (1.3) is of signature $(3,1)$ over $\mathbb{R}$, we have that A is a subgroup of $\mathrm{O}(3,1)$ and can be thought of as a subgroup of the group of motions of hyperbolic 3-space $\mathbb{H}^3$. In this way A is a discrete algebraic group acting on $\mathbb{H}^3$ where the complement of three mutually tangent hemispheres inside an infinite cylinder is the fundamental domain of the action. This fundamental domain has infinite volume, which makes counting integers in the group's orbit quite difficult. We

[2]Note that the curvatures in any packing are determined by the structure of the group A rather than the vector $\mathbf{v}$ itself, and considering an ACP as an orbit of the group allows us to work with *all* ACP's at once.

note, however, that A contains Fuchsian triangle subgroups generated by any three of the S_i above, which are lattices in the corresponding $\mathrm{O}(2,1)$'s. We use this fact extensively throughout this paper.

To this end, denote by A_i the subgroup of A generated by three of the four generators as follows:

$$A_i := (\{S_1, S_2, S_3, S_4\} - \{S_i\}).$$

This group is the Schottky group generated by reflections in the three circles intersecting the ith circle in the root quadruple and perpendicular to the initial circles in the packing; in particular, the ith circle is fixed under this action. The fundamental domain of A_i is then a triangle bounded by the three circles, and has hyperbolic area π.

2. A PRELIMINARY LOWER BOUND

In this section, we follow [S2] in order to count integer points in an orbit of a subgroup A_i of the Apollonian group as described in Section 1.1. This produces a preliminary lower bound on the number $\kappa(P,X)$ of integers less than X occurring as curvatures in an Apollonian packing P.

Proposition 2.1. *For an integer Apollonian circle packing P, let $\kappa(P,X)$ denote the number of distinct integers less than X occurring as curvatures in the packing. Then we have*

$$\kappa(P,X) \gg \frac{X}{\sqrt{\log X}}.$$

Proof. We fix a circle C_0 of non-zero curvature a_0 in the packing P, and count the integers which occur as curvatures of circles tangent to C_0 in P. This is identical to considering the orbit of A_1 acting on a quadruple $\mathbf{v}$ of mutually tangent circles (a_0, b, c, d), since A_1 fixes the first coordinate of $\mathbf{v}$ and its orbit represents all of the circles tangent to C_a. Note that A generates all possible Descartes configurations in the packing P, and there can only be finitely many circles of curvature a_0 in the packing since the total area of all the inscribed circles is bounded by the area of the outside circle. Therefore it is reasonable to count the circles represented in the orbit of A_1, since they make up a positive fraction of all of the circles in P tangent to a circle of curvature a_0.

In this orbit, we have that the first coordinate a_0 is fixed, and the other coordinates of points in the orbit of A_1 vary to satisfy

$$Q(a_0, x_2, x_3, x_4) = 2(a_0^2 + x_2^2 + x_3^2 + x_4^2) - (a_0 + x_2 + x_3 + x_4)^2 = 0,$$

where Q is the Descartes form in (1.3). A change of variables $\mathbf{y} = (y_2, y_3, y_4) = (x_2, x_3, x_4) + (a_0, a_0, a_0)$ allows us to rewrite the equation above as

$$g(\mathbf{y}) + 4a_0^2 = 0, \tag{2.1}$$

where $g(\mathbf{y}) = y_2^2 + y_3^2 + y_4^2 - 2y_2y_3 - 2y_2y_4 - 2y_3y_4$ is the resulting ternary quadratic form. We can thus conjugate the action of A_1 on (a_0, x_2, x_3, x_4) to an action independent of a_0 which preserves the form g. This is the action of a group Γ on $\mathbf{y}$, generated by

$$\begin{pmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}.$$

Moreover, the action of Γ on

$$\mathbf{v}' = (b + a_0, c + a_0, d + a_0)$$

is related to the action of A_1 on $\mathbf{v}$ by

$$A_1\mathbf{v} = (a_0, \Gamma[\mathbf{v}' - (a_0, a_0, a_0)]),$$

so we count the same number of curvatures occurring in the packing before and after this change of variables. We change variables once again by letting

$$y_2 = A,\ y_3 = A + C - 2B,\ y_4 = C.$$

We note that $(y_2, y_3, y_4) \in \mathbb{Z}^3$ implies that A, B, and C are integers, and the primitivity of the packing is preserved as well – the gcd of A, B, and C is 1. With this change of variables, Γ is conjugated to an action of a group Γ' on (A, B, C) which is generated by

$$\begin{pmatrix} 1 & -4 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{pmatrix}.$$

Under this change of variables, the expression in (2.1) becomes

$$4(B^2 - AC) = -4a_0^2. \tag{2.2}$$

Letting $\Delta(A, B, C)$ denote the discriminant of the binary quadratic form $Ax^2 + 2Bxy + Cy^2$, (2.2) is simply

$$\Delta(A, B, C) = a_0^2,$$

and thus Γ' is a subgroup of $\mathrm{O}_\Delta(\mathbb{Z})$, the orthogonal group preserving Δ. Let $\tilde{\Gamma}$ denote the intersection $\Gamma' \cap \mathrm{SO}_\Delta(\mathbb{Z})$. The spin double cover of SO_Δ is well known (see [EGM]) to be PGL_2, and is obtained via the homomorphism

$$\begin{array}{rcl} \rho : \mathrm{PGL}_2(\mathbb{Z}) & \longrightarrow & \mathrm{SO}_\Delta(\mathbb{Z}) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \overset{\rho}{\longmapsto} & \frac{1}{\alpha\delta-\beta\gamma} \cdot \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta+\beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\delta & \gamma^2 \end{pmatrix} \end{array} \tag{2.3}$$

written over $\mathbb{Z}$ as this is the situation we work with. It is natural to ask for the preimage of $\tilde{\Gamma}$ under ρ which we determine in the following lemma.

Lemma 2.2. *Let $\tilde{\Gamma}$ and ρ be as before. Let $\Lambda(2)$ be the congruence 2-subgroup of $PSL_2(\mathbb{Z})$. Then the preimage of $\tilde{\Gamma}$ in $PGL_2(\mathbb{Z})$ under ρ is $\Lambda(2)$.*

Proof. We can extract from the generators of Γ' as well as the formula in (2.3) that the preimage of $\tilde{\Gamma}$ under ρ contains

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

and so $\tilde{\Gamma}$ contains the congruence subgroup $\Lambda(2)$ of $\mathrm{SL}_2(\mathbb{Z})$. Recall that the area of $A_1\backslash\mathbb{H}^2$ is π, and note that $\mathrm{SO}_\Delta(\mathbb{Z})\bigcap\Gamma'$ contains exactly those elements of Γ' which have even word length when written via the generators of Γ', making up half of the whole group. Therefore the area of $\tilde{\Gamma}\backslash\mathbb{H}^2$ is 2π, which is equal to the area of $\Lambda(2)\backslash\mathbb{H}^2$, and hence the preimage of $\tilde{\Gamma}$ in $\mathrm{PGL}_2(\mathbb{Z})$ is precisely $\Lambda(2)$ as desired. $\square$

Recall that we would like to count the integer values of y_2, y_3, and y_4 – in terms of the action of Γ, we are interested in the set of values A, C, and $A+C-2B$ above. Lemma 2.2 implies that these values contain integers represented by the binary quadratic form

$$f_{a_0}(\zeta,\nu) = A_0\zeta^2 + 2B_0\zeta\nu + C_0\nu^2, \tag{2.4}$$

where $(\zeta,\nu)=1$, and the coefficients are derived from the change of variables above:

$$A_0 = b + a_0,\ C_0 = d + a_0,\ B_0 = \frac{b+d-c}{2}.$$

We note that the discriminant of this form is not a square, since

$$D(f_{a_0}) = (2B_0)^2 - 4A_0C_0 = -4a_0^2 \tag{2.5}$$

and $a_0 \neq 0$. Since the vectors in the orbit of A_1 are of the form $(a_0, A-a_0, A+C-2B-a_0, C-a_0)$, they correspond to the integer values of

$$f_{a_0}(x,y) - a_0, \tag{2.6}$$

where f_{a_0} is as before. Therefore

$$\kappa(P,X) \gg \#\{m \in \mathbb{Z} \,|\, m > 0,\, f_{a_0}(x,y) - a_0 = m \text{ for some } x, y \in \mathbb{Z}, (x,y) = 1\} \tag{2.7}$$

and we need only to count the integers represented by f_{a_0} in order to get a bound on the number of curvatures in P. To this end we have the following lemma of Blomer from [B] regarding the number of integers less than x represented by a binary quadratic form of discriminant D.

Lemma 2.3. *(Blomer) Let f be a positive definite binary quadratic form over $\mathbb{Z}$ of discriminant $-D$, where D is a positive integer. Let $U_f(X)$ be the number of positive integers not exceeding X representable by f and let $\tau = \log 2$. Then we have*

$$U_f(X) \gg_\varepsilon \frac{X}{|D|^\varepsilon \sqrt{\log X}}$$

uniformly in $|D| \le (\log X)^{\tau-\varepsilon}$, and

$$U_f(X) \gg_\varepsilon \frac{X}{(\log X)^{\tau+\varepsilon}}$$

uniformly in $|D| \le (\log X)^{2\tau-\varepsilon}$.

An analog of this was first proven for sums of two squares by Landau in [L2], and was generalized to binary quadratic forms of non-square discriminant by James in [J]. We use Blomer's result because it explains how $U_f(X)$ depends on the size of the discriminant of f. Since the number of integers represented by the form f_{a_0} of discriminant $D = -4a^2$ is a lower bound for the number of integers occurring as curvatures in the orbit of A_1, the value of $U_{f_{a_0}}(x)$ as written in Blomer's lemma is a lower bound for our count, and we have

$$\kappa(P,X) > \frac{cX}{\sqrt{\log X}}$$

as desired. □

We use this preliminary lower bound several times to prove Theorem 1.1 in the next section.

3. Proof of Theorem 1.1

In this section, we repeat the method in Section 2 to improve on the bound calculated in Proposition 2.1. Recall that to prove Proposition 2.1, we counted only the circles tangent to some fixed circle C_0 in the packing. Denote the curvatures less than X found in this way by $\mathscr{A}_0$:

$$\mathscr{A}_0 = \{a \le X \,|\, a > 0, a = f_{a_0}(x,y) - a_0 \text{ for some non-negative } x, y \in \mathbb{Z}\}$$

where f_{a_0} is the binary quadratic form defined in (2.4) and each $a \in \mathscr{A}_0$ corresponds to the curvature of a circle C_a in the packing P. To prove Theorem 1.1 we carry out the analysis in Section 2 several times, fixing a different circle C_a to generate a new binary quadratic form $f_a - a$ for $a \in \mathscr{A}_0$ each time. In this way, we obtain a set

$$H_a = \{\gamma = f_a(\zeta, \nu) - a \,|\, \gamma \leq X\}$$

for each $a \in \mathscr{A}_0$ whose cardinality is given by Lemma 2.3. Since we seek to count curvatures of circles without multiplicity, we must also keep track of how many integers we count twice. To do this, it is important to note that the integers represented by f_a and $f_{a'}$ for $a \neq a' \in \mathscr{A}_0$ are a subset of integers which can be written as a sum of two squares since both forms have discriminant of the form $-\delta^2$. In fact, f_a and $f_{a'}$ represent practically the same integers – it is rather the *shift* of each form f_a by a in (2.6) that makes the resulting integers in H_a vary significantly and allows us to compute an upper bound for

$$|H_a \cap H_{a'}|$$

using an old theorem of Rieger in [R].

3.1. **Counting curvatures by fixing circles tangent to C_0.** We associate a circle C_a tangent to C_0 to every $a \in \mathscr{A}_0$ and produce a shifted binary quadratic form

$$f_a(\zeta, \nu) - a \tag{3.1}$$

via an appropriate action of a Fuchsian subgroup A_i of A paired with the change of variables in Section 2. Since the integers represented by this form once again make up a subset of integers arising as curvatures of circles tangent to C_a, our goal is to count these integers without multiplicity. To this end, we have the following lemma regarding integers represented by forms f_a where $a \leq (\log X)^{(\log 2)/2}$.

Lemma 3.1. *Let f_a be as before, let $\tau = \log 2$ and let*

$$H_a = \{\gamma = f_a(\zeta, \nu) - a \,|\, \gamma \leq X\}$$

We have

$$\sum_{\substack{a \in \mathscr{A}_0 \\ a \leq (\log X)^{\frac{\tau}{2}}}} |H_a| > \frac{cX}{(\log X)^{\varepsilon}},$$

where $\varepsilon = \frac{1-\tau}{2} = 0.153\ldots$ and c depends on a_0 only.

Proof. Denote by $r_a(n)$ the number of times n is represented by the form in (3.1). Since the discriminant $D(f_a) \leq (\log X)^\tau$ for $a \leq (\log X)^{\frac{\tau}{2}}$, there exists $c' > 0$ such that

$$|H_a| = \sum_{n \leq X} r_a^0(n) > \frac{c'X}{\sqrt{\log X}}$$

by Lemma 2.3. We also have

$$\#\{a \in \mathscr{A}_0 \,|\, a \leq (\log X)^{\frac{\tau}{2}}\} > \frac{c''(\log X)^{\frac{c}{2}}}{\sqrt{\log\log X}}$$

for some $c'' > 0$ by the same lemma. Therefore

$$\begin{aligned}
\sum_{\substack{a \in \mathscr{A}_0 \\ a \leq (\log X)^{\frac{\tau}{2}}}} |H_a| &= \#\{a \in \mathscr{A}_0 \,|\, a \leq (\log X)^{\frac{\tau}{2}}\} \cdot |H_a| \\
&> \frac{c''' \cdot \sqrt{2}X}{\sqrt{\log\log X}(\log X)^{\frac{1-\tau}{2}}} \\
&> \frac{cX}{(\log X)^{\frac{1-\tau}{2}}} \qquad (3.2)
\end{aligned}$$

as desired. $\square$

It remains to determine how many integers we counted more than once in this way.

3.2. **Counting integers in the intersections.** Since we must count the number of integers in the union

$$\bigcup_{\substack{a \in \mathscr{A}_0 \\ a \leq (\log X)^{\frac{\tau}{2}}}} |H_a|$$

to prove Theorem 1.1, we must compute an upper bound on the size of the intersections

$$H_a \cap H_{a'}$$

for $a \neq a \in \mathscr{A}_0$, and $a, a' \leq (\log X)^{\frac{\tau}{2}}$. We do this in the following lemma.

Lemma 3.2. *Let $\mathscr{A}_0$ and H_a be as before. We have*

$$\sum_{\substack{a \neq a' \in \mathscr{A}_0 \\ a, a' \leq (\log X)^{\frac{\tau}{2}}}} |H_a \cap H_{a'}| \leq \frac{cX}{\log\log X}$$

where c depends on a_0 only.

Note that $|H_a \cap H_{a'}|$ for $a \neq a'$ is bounded above by the number of times $b = a' - a$ is represented by the quaternary form

$$f_a(x,y) - f_a(x',y'),$$

and that, since the discriminants of the f_a's we consider are all of the form δ^2, we have

$$H_a \subset \{n \in \mathbb{Z} \,|\, n = \alpha^2 + \beta^2 \text{ for some } \alpha, \beta \in \mathbb{Z}\}$$

for any $a \in \mathscr{A}_0$. Therefore we have that

$$|H_a \cap H_{a'}| \leq |\{n \in \mathbb{Z} \,|\, n = \alpha^2 + \beta^2 \text{ and } n + b = \alpha'^2 + \beta'^2 \text{ for some } \alpha, \beta, \alpha', \beta' \in \mathbb{Z}\}|$$

In the case of sums of two squares, bounds on the number of pairs of integers separated by a gap b and representable as a sum of two squares is well known. In [I], Indlekofer calculates this for $b = 1$. He proves that if $B(X)$ is the set of integers less than X and representable as a sum of two squares and

$$B_1(X) := \{n \in B(X) |\, n+1 \in B(X)\},$$

then $|B_1(X)|$ is a constant multiple of $\frac{X}{\log X}$. For our purposes, we are interested in pairs of integers separated by an arbitrary gap b representable as sums of two squares, which we denote by

$$B_b(X) = \{n \in B(X) |\, n+b \in B(X)\}. \tag{3.3}$$

In [R], Rieger proves the following lemma about $|B_b(x)|$:

Lemma 3.3. *Let $B_b(X)$ be as above. Then we have that*

$$|B_b(X)| \ll \frac{X}{\log X} \prod_{\substack{p|b \\ p \equiv 3\ (4)}} \left(1 + \frac{1}{p}\right).$$

Since the integers less than X represented by the forms f_a are a subset of the integers in $B(X)$ above, we are able to use the upper bound in Rieger's lemma to count the integers occurring in $H_a \cap H_{a'}$.

Proof of Lemma 3.2:

As before, we note that for every $n < X$ such that $n \in B(X)$ and $n + (a' - a) \in B(x)$ there is a point $(\alpha, \beta, \alpha', \beta')$ such that

$$(f_{a'}(\alpha', \beta') - a') - (f_a(\alpha, \beta) - a) = f_{a'}(\alpha', \beta') - f_a(\alpha, \beta) - (a' - a) = 0$$

Thus we have

$$|H_a \cap H_{a'}| \leq |B_b(X)|,$$

where $B_b(X)$ is as in (3.3). Lemma 3.3 implies

$$|B_b(X) \ll \frac{X}{\log X} \prod_{\substack{p|b \\ p\equiv 3\ (4)}} \left(1+\frac{1}{p}\right)$$

for $b = |a' - a|$. From (**??**), we have that the number of pairs $(f_a, f_{a'})$ considered in Section 3.1 is at most

$$\frac{(\log x)^\tau}{\log\log x},$$

where $\tau = \log 2$. We use this to compute the number of repeated integers counted in Step 2 below:

$$\begin{aligned}\sum_{\substack{a\neq a'\in\mathscr{A}_0 \\ a,a'\leq(\log X)^{\frac{\tau}{2}}}} |H_a \cap H_{a'}| &\ll \frac{X}{\log\log X} \max_{\substack{a\neq a'\in\mathscr{A}_0 \\ a,a'\leq(\log X)^{\frac{\tau}{2}}}} \left(\prod_{\substack{p||a-a'| \\ p\equiv 3\,(4)}} \left(1+\frac{1}{p}\right)\right) \\ &\ll \frac{X}{\log\log X}\end{aligned}$$

as desired. $\square$

Combined with Lemma 3.1, we have

$$\kappa(P,X) > \frac{c'X}{(\log X)^{\tau'}} - \frac{c''X}{\log\log X} \sim \frac{cX}{(\log X)^{\tau'}}$$

where $\tau' = \frac{1-\log 2}{2} = 0.153\ldots$ and c is a constant depending on P as desired.

While this method is promising in attempting to prove a result towards positive density, it would require some careful analytic fine-tuning in each step in order to perhaps produce the result conjectured in [GLMWY].

APPENDIX

In this section we follow [L1] to clarify which integers are represented by the binary quadratic forms f_a described in Section 2. This is meant to clarify why forms f_a and $f_{a'}$ represent approximately the same integers less than X for $a \neq a'$ as in the previous sections. We begin by noting that, given an integer γ represented by some symmetric binary quadratic form of determinant $-\delta^2$, if $p^r||\gamma$ and p is congruent to 3 mod 4, we must have that the maximal power r of p dividing γ is even.

Lemma 3.4. *Let $f_a(x,y) = Ax^2 + 2Bxy + cy^2$, with the discriminant $B^2 - AC = -\delta^2$ for $\delta > 0$. Suppose $n \in \mathbb{Z}$ is represented by f_a, and let $p|n$ denote a prime not dividing δ, with $p \cong 3\ (4)$. Then we have that $n = n'p^{2v}$, where $p \not| n'$ and v is an integer.*

Proof. We diagonalize the form $f_{a'}$ to

$$F(x,y) = A'x^2 + C'y^2,$$

where we know $A'C' = a^2$. Let p be a prime as above. Since $p \nmid \delta$, we have p does not divide either A' or C'. Now suppose

$$A'x^2 + C'y^2 \equiv 0 \ (p^w),$$

for some pair of integers (x,y). If $p|x$ and $p|y$, we have that w must be even, and we are done.

Suppose this is not the case. Then without loss of generality we have that $p \nmid y$, and there exists an integer z such that

$$yz \equiv 1 \ (p^w),$$

and so we have

$$A'(xz)^2 + C'(yz)^2 \equiv 0 \ (p^w),$$

or

$$A'(xz)^2 \equiv -C' \ (p^w).$$

Since $p \nmid A'$, there is an integer $(A')^{-1}$ such that $(A')^{-1}A' \equiv 1 \ (p)$, and we rewrite the equation above as

$$(xz)^2 \equiv -(A')^{-1}C' \ (p^w). \tag{3.4}$$

Since $A'C' = a^2$, we have that $-(A')^{-1}C' = -((A')^{-1})^2 a^2$. This, together with (3.4), implies -1 is a square modulo p^w, and so w must be even. □

With Lemma 3.4 in mind, we now specify exactly which integers are represented by the form f_a.

Proposition 3.5. *Let f_a be a binary quadratic form as before, with discriminant $-\delta^2$. Then if $f_a(x,y) = N$ for some integers x and y where N is relatively prime to a, we have that can be written as $N = v^2 n$, where n is square free and has no prime divisors congruent to* 3 *modulo* 4.

Proof. We begin, again, by diagonalizing our form to

$$F(x,y) = A'x^2 + C'y^2.$$

Since n has no prime divisors congruent to 3 mod 4, we know that $\left(\frac{-1}{n}\right) = 1$. In particular, this means that

$$-\delta^2 \equiv w^2 \ (n)$$

for some integer w. We rewrite this as

$$nk - \delta^2 = w^2, \text{ or}$$

$$nk - w^2 = \delta^2.$$

So the binary form $F'(x,y) = nx^2 + wxy + ky^2$ has discriminant $-\delta^2$, and is equivalent to $F(x,y)$. Since F' represents n in the obvious way, we have that F represents it as well. Thus F will represent any integer of the form v^2n above. □

Note that the integers represented by f_a in Proposition 3.5 depend minimally on a – the only constraint is that the integers represented by f_a have to be relatively prime to a. Thus a binary quadratic form of discriminant $-a^2$ will represent almost the same integers as a binary quadratic form of discriminant $-a'^2$, the difference coming only from the prime divisors of a and a', and we have what we want.

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