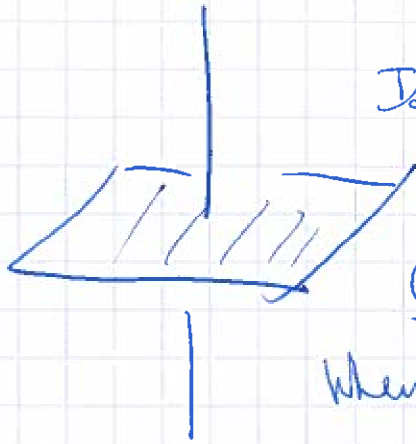
Lecture 2

Ex: $Z = \{x=y=0\} \cup \{z=0\}$



Ideal $= (x,y) \cap (z) \subset \mathbb{K}[x,y]$

~~[illegible]~~

Claim $(x,y) \cap (z) = (xz, yz)$

When $g = xg_1 + yg_2$ is divisible by z?

- $g \in (x,y)$ if all monomials in g are divisible by x or y
- $g \in (z)$ if all monomials in g are divisible by z

$\Longleftrightarrow$ all monomial are divisible by xz or yz. $\square$

Conclusion: $Z = \{xz = yz = 0\}$.

Def $I \subset \mathbb{K}[x_1 \dots x_n]$ ideal. Then

$$\sqrt{I} = \{ f \in \mathbb{K}[x_1 \dots x_n] : f^N \in I \text{ for some } N \}.$$

Lemma a) $\sqrt{I}$ is an ideal

b) $\sqrt{\sqrt{I}} = \sqrt{I}$

Proof: b) is clear

a) $f, g \in \sqrt{I}$ $\quad f^{N_1} \in I \quad g^{N_2} \in I$

we claim that $(f+g)^{N_1+N_2} \in I \Rightarrow f+g \in \sqrt{I}$

Indeed, $(f+g)^{N_1+N_2} = \sum_{a+b=N_1+N_2} \binom{N_1+N_2}{a} f^a g^b$

In each term either $a \geq N_1$ or $b \geq N_2 \Rightarrow$ all terms in I.

$f \in \sqrt{I}$, g arbitrary $\Rightarrow (fg)^N = f^N g^N \in I$
$f^N \in I$.
So $\sqrt{I}$ is an ideal.

Lemma $Z(\sqrt{I}) = Z(I)$

Proof: $I \subset \sqrt{I} \Rightarrow Z(\sqrt{I}) \subset Z(I)$
$f \in \sqrt{I}$, $p \in Z(I) \Rightarrow f^N(p) = 0 \Rightarrow f(p) = 0$.
So $p \in Z(\sqrt{I})$.

Ex $I = (x^2, xy, xz, yz)$ $\quad \sqrt{I} = (x, yz)$ Prove it!

Proof: $\sqrt{I}$ contains x, yz. Suppose $f \in \sqrt{I}$. By subtracting all terms divisible by x, we can assume $f = f(y,z)$.
Now $f(y,z)^N \in I$ if $f(y,z)^N$ divisible by yz
$\Rightarrow f(y,z)$ is divisible by yz.

Thm 1 (Hilbert's Nullstellensatz, part 1)
Suppose $\mathbb{K}$ is algebraically closed. Then all maximal ideals in $\mathbb{K}[x_1 \dots x_n]$ are of the form
$(x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1 \dots a_n) \in \mathbb{A}^n$.
Proof: Friday

Thm 2 (Nullstellensatz, part 2) Suppose $\mathbb{K}$ is algebraically closed and $I \subset \mathbb{K}[x_1 \ldots x_n]$ is a proper ideal. Then $Z(I) \neq \emptyset$.

Thm 3 (Nullstellensatz, part 3) Suppose $\mathbb{K}$ is alg. closed, I = ideal, f vanishes at all points of $Z(I)$. Then $f^N \in I$ for some N, so $f \in \sqrt{I}$.

Remarks:

① The ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal.

Recall: I is maximal $\Leftrightarrow$ $\mathbb{K}[x_1 \ldots x_n]/I$ is a field.

Need to compute $\mathbb{K}[x_1 \ldots x_n] / (x_1 - a_1, \ldots x_n - a_n)$.

Change coordinates: $y_i = x_i - a_i$, we can rewrite any polynomial in x_i as a polynomial in y_i, and vice versa.

Now $\dfrac{\mathbb{K}[y_1 \ldots y_n]}{(y_1 \ldots y_n)} \simeq \mathbb{K}$ since $(y_1 \ldots y_n)$ contains all polynomials in y_i with zero constant terms.

② More conceptually, we have a map

$\mathbb{K}[x_1 \ldots x_n] \xrightarrow{F} \mathbb{K}$ "evaluation map"

$$F(f) = f(a_1 \ldots a_n)$$

$\mathrm{Im}\, F = \mathbb{K}$ $\quad$ $\mathrm{Ker}\, F = \{$all polynomial f such that $f(a_1 \ldots a_n) = 0\}$

$\dfrac{\mathbb{K}[x_1 \ldots x_n]}{\mathrm{Ker} F} = \mathrm{Im}\, F = \mathbb{K}$. $\qquad = (x_1 - a_1, \ldots, x_n - a_n)$

③ Theorems 1 and 2 are false if $\mathbb{K}$ is not algebraically closed. For example, $\mathbb{K} = \mathbb{R}$ then the ideal $(x^2+1) \subset \mathbb{K}[x]$ is maximal.

Proof: $\mathbb{K}[x]$ is a PID so any ideal is principal (f)

$\mathbb{K}[x] \supset (f) \supset (x^2+1) \Rightarrow f$ divides x^2+1

But x^2+1 is irreducible, contradiction.

Similarly $Z(x^2+1) = \emptyset$.

$\boxed{\text{Thm } 1 \Rightarrow \text{Thm } 2}$ Suppose $I \subset \mathbb{K}[x_1 \dots x_n]$ is a proper ideal. Then there exists a maximal ideal m such that $\mathbb{K}[x_1 \dots x_n] \supset m \supset I$.

By Thm 1 $m = (x_1 - a_1, \dots x_n - a_n)$ so

$I \subset (x_1 - a_1, \dots, x_n - a_n)$.

Therefore all polynomials in I vanish at $(a_1 \dots a_n)$

$\Rightarrow (a_1 \dots a_n) \in Z(I)$.