Lecture 3 | **Thm** Assume $\mathbb{K}$ alg. closed. Then all maximal ideals in $\mathbb{K}[x_1 \_ x_n]$ are of the form $(x_1 - a_1, \ldots x_n - a_n)$.

Before proving this, we need to review some commutative algebra

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Def $K \subset L$ rings. We say that $z \in L$ is *algebraic* over K if $z^n + a_1 z^{n-1} + \ldots + a_n = 0$ for $a_i \in K$.

Fact Algebraic elements of L form a *ring*.

Lemma K = field, $z \in K(x)$ and

$$z^n + a_1 z^{n-1} + \ldots + a_n = 0 \quad \text{for } a_i \in K[x].$$

Then $z \in K[x]$.

Proof $z = \frac{f}{g}$, $GCD(f, g) = 1$

$$\left(\frac{f}{g}\right)^n + a_1 \left(\frac{f}{g}\right)^{n-1} + \ldots + a_n = 0$$

$$\underbrace{f^n}_{\text{coprime to } g} + \underbrace{a_1 f^{n-1} g + \ldots + a_n g^n}_{\text{divisible by } g} = 0$$

$\Rightarrow g = 1$. $\square$

Thm $K \subset L$ fields, and L is finitely generated over K

(1) That is, there exist $v_1, \ldots v_n \in L$ such that

any element of L can be written as a polynomial

in $v_1 \ldots v_n$ with coef in K, then any elt of L

is algebraic over K.

<u>Rmk</u> L is finitely generated $\Leftrightarrow$ there is a surjective

homomorphism $K[x_1 \ldots x_n] \twoheadrightarrow L \quad x_i \mapsto v_i$

<u>Pf of Thm</u>: Induction in n

$n=1$ $K[x] \overset{\varphi}{\twoheadrightarrow} L$ surjective, $x \mapsto v_1$

Recall, $K[x]$ is not a field, PID

- If $\operatorname{Ker}\varphi = 0 \Rightarrow L = K[x]$ not a field
  contradict.

So $\operatorname{Ker}\varphi = (f)$ for some $f \neq 0$
ideal in $K[x]$

Therefore $L = \dfrac{K[x]}{(f)}$ and L is algebraic.

Indeed, x is algebraic in $\dfrac{K[x]}{(f)}$, algebraic elements

form a subring ~~containing~~ containing $x \Rightarrow \dfrac{K[x]}{(f)}$.

$n>1$ L is generated by $v_2 \ldots v_n$ over $K(v_1)$
field

$\Rightarrow$ by assumption of induction L is algebraic

over $K(v_1)$

Case 1 v_i is algebraic over $K \Rightarrow$ we are done

Fact $\underbrace{K \subset K(x)}_{\text{algebraic}} \underbrace{\subset L}_{\text{algebr.}} \Rightarrow \underbrace{K \subset L}_{\text{algebraic}}$.

Case 2 v_1 is <u>not</u> algebraic over $K \Rightarrow v_i$ $i \geq 2$ algebraic over $K(v_1)$ is $K(x)$

$$v_i^{n_i} + a_{i1} v_i^{n_i - 1} + \dots + a_{in_i} = 0$$

$a_{ij} \in K(v_1)$

a = common denominator of all a_{ij}

$$(av_i)^{n_i} + a \cdot a_{i1} (av_i)^{n_i - 1} + \dots + a^{n_i} a_{in_i} = 0$$

$\Rightarrow av_i$ is algebraic over $K[v_1]$.

If $z \in L$ then z is a polynomial in $v_1 \dots v_n$

$\Rightarrow a^N z$ is a polynom in $v_1, av_2, \dots av_n$

$\Rightarrow a^N z$ is algebraic over $K[v_1]$

Pick $z = \frac{1}{c}$, $c \in K[v_1]$ coprime to a

can do this since infinitely many irreducible polynomials $K(x)$

$\frac{a^N}{c} \in K(v_1)$ algebraic over $K[v_1]$

By <u>Lemma</u> $\Rightarrow c = 1$ contradiction.

Thm $\mathbb{K}$ alg. closed $\Rightarrow$ all maximal ideals in $\mathbb{K}[x_1 \ldots x_n]$ are $(x_1 - a_1, \ldots x_n - a_n)$

Pf: $\mathfrak{m}$ = max. ideal in $\mathbb{K}[x_1 \ldots x_n]$

$$L = \mathbb{K}[x_1 \ldots x_n]/\mathfrak{m} = \text{field}.$$

By construction, L is finitely generated over $\mathbb{K}$
$\Rightarrow$ by Thm L is algebraic over $\mathbb{K}$.
But $\mathbb{K}$ is alg. closed $\Rightarrow$ $L = \mathbb{K}$.
$x_i \Longleftrightarrow a_i$ in L for some $a_i \in \mathbb{K}$
$\Rightarrow x_i - a_i \in \mathfrak{m} \Rightarrow (x_1 - a_1, \ldots x_n - a_n) \subset \mathfrak{m}$.
But $(x_1 - a_1, \ldots x_n - a_n)$ is maximal, so
$\mathfrak{m} = (x_1 - a_1, \ldots x_n - a_n)$.

Thm We have a bijection: ($\mathbb{K}$ alg. closed) (I = ideal in $\mathbb{K}[x_1 \ldots x_n]$)

$$\left\{\begin{matrix}\text{maximal} \\ \text{ideals in } \mathbb{K}[x_1 \ldots x_n]/I\end{matrix}\right\}_{①} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals} \\ \text{in } \mathbb{K}[x_1 \ldots x_n] \text{ containing } I\end{matrix}\right\}_{②}$$

$$② \longleftrightarrow \left\{\text{points in } Z(I)\right\}_{③}$$

Pf ① $\Leftrightarrow$ ② exercise

② $\Leftrightarrow$ ③ By Nullstellensatz, ② = ideals $(x_1 - a_1, \ldots x_n - a_n)$.
This contains $I \Leftrightarrow$ all function from I vanish at $(a_1 \ldots a_n) \Leftrightarrow (a_1 \ldots a_n) \in Z(I)$