lecture 4 Nullstellensatz and its consequences

$\mathbb{K}$ alg. closed

Thm 1 All maximal ideals in $\mathbb{K}[x_1 \dots x_n]$ are of the form $(x_1-a_1, x_2-a_2, \dots x_n-a_n)$ [Proved Friday]

Thm 2 $I \subset \mathbb{K}[x_1 \dots x_n]$ proper ideal $\Rightarrow Z(I) \neq \emptyset$

Thm 3 $I \subset \mathbb{K}[x_1 \dots x_n]$ some ideal, $f(x_1 \dots x_n)$ vanishes at all points of $Z(I) \Rightarrow f \in \sqrt{I}$.

Proof of Thm 3 Define $J \subset \mathbb{K}[x_1 \dots x_n, x_{n+1}]$

$$J = I[x_{n+1}] + (x_{n+1} f - 1)$$

Let us prove that $Z(J) = \emptyset$. Indeed, suppose $(a_1 \dots a_{n+1}) \in Z(J)$, then

$(a_1 \dots a_n) \in Z(I)$ and $a_{n+1} \cdot f(a_1 \dots a_n) = 1$

$\Downarrow$

$f(a_1 \dots a_n) = 0$ contradiction.

By Thm 2 we get $J = \mathbb{K}[x_1 \dots x_n, x_{n+1}] \Rightarrow 1 \in J$

$$1 = \sum g_i(x_1 \dots x_n) x_{n+1}^i + h(x_1 \dots x_n, x_{n+1})(x_{n+1} f - 1)$$

Plug in $x_{n+1} = \frac{1}{f}$, get identity of rational functions

$$1 = \sum g_i(x_1 \dots x_n)\left(\frac{1}{f}\right)^i + \cancel{h \cdot 0}$$

Clear denominators: $f^N = \sum g_i(x_1 \dots x_n) f^{N-i} \in I$.

Y = subset of $\mathbb{A}^n$

<u>Def</u> $I(Y) = \{ f \in \mathbb{K}[x_1 \dots x_n] \mid f(x_1 \dots x_n) = 0$ for all points $(x_1 \dots x_n) \in Y \}$

<u>Lemma</u> $I(Y)$ is a radical ideal

<u>Proof</u>: • $f, g \in I(Y)$. f, g vanish at all point $p \in Y$
$\Rightarrow f(p) + g(p) = 0 \Rightarrow f+g \in I(Y)$

• $f \in I(Y)$, g arbitrary $\Rightarrow f(p) g(p) = 0$ for $p \in Y$

$\Rightarrow I(Y)$ is an <u>ideal</u>.

• $f^N \in I(Y) \Rightarrow f^N(p) = 0$ for all $p \in Y \Rightarrow f(p) = 0$
$\Rightarrow f \in I(Y)$.

$\square$

<u>Lemma</u> (a) If $Y_1 \subset Y_2$ then $I(Y_1) \supset I(Y_2)$

(b) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(c) Suppose J is an ideal in $\mathbb{K}[x_1 \dots x_n]$
then $I(Z(J)) = \sqrt{J}$.

In particular, if J is <u>radical</u> then $I(Z(J)) = J$.

<u>Proof</u> (a) If $f(p) = 0$ for all $p \in Y_2$ then $f(p) = 0$ for all $p \in Y_1$

(b) Clear / exercise

(c) This is <u>Thm 3</u> (Nullstellensatz).

Def Given $Y \subset \mathbb{A}^n$, we define $\overline{Y}$ as the smallest algebraic set containing Y.

Lemma (a) $I(\overline{Y}) = I(Y)$ (b) $Z(I(Y)) = \overline{Y}$

Proof: (a) $Y \subset \overline{Y} \Rightarrow I(\overline{Y}) \subset I(Y)$. Conversely, suppose $f \in I(Y)$. Then $\{f=0\}$ is an algebraic set contains Y. Recall that intersection of alg sets is algebraic, so $\{f=0\} \cap \overline{Y}$ is algebraic set contain $Y \Rightarrow \{f=0\} \cap \overline{Y} = \overline{Y} \Rightarrow \overline{Y} \subset \{f=0\}$. Therefore f vanishes at all points of $\overline{Y}$.

(b) $Z(I(Y))$ = alg. set containing $Y \Rightarrow$

$Z(I(Y)) \supset \overline{Y}$. Conversely, since $\overline{Y}$ is algebraic we can write $\overline{Y} = Z(J)$ for some ideal J. Then

$$I(Y) \underset{(a)}{=} I(\overline{Y}) = I(Z(J)) \overset{\text{Nullstellensatz}}{=} \sqrt{J}$$

$$Z(I(Y)) = Z(\sqrt{J}) = Z(J) = \overline{Y}. \qquad \square$$

Thm Assume $\mathbb{K}$ alg. closed. There is an inclusion-reversing bijection

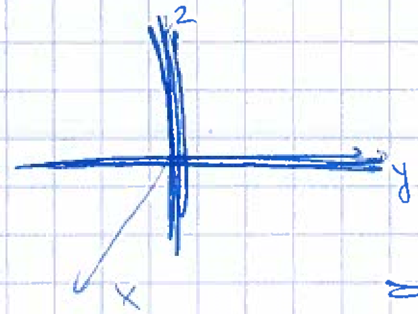
$$\left\{ \begin{array}{c} \text{algebraic subsets} \\ \text{of } \mathbb{A}^n \end{array} \right\} \underset{Z}{\overset{I}{\rightleftarrows}} \left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } \mathbb{K}[x_1 \dots x_n] \end{array} \right\}$$

Cor $\sqrt{I}$ = unique radical ideal J such that $Z(I)=Z(J)$.

Ex $I=(x^2, xy, xz, yz)$ (compare with lecture 2)

$Z(I) = \{x^2 = xy = xz = yz = 0\}$

$= \{x = 0 = yz\}$



$\downarrow$ 2 lines

(x, yz) is radical (prove it!)

$\Rightarrow (x, yz) = \sqrt{I}$.
