Lecture 5 | **Def** An ideal $p \subset \mathbb{K}[x_1 \dots x_n]$ is <u>prime</u> if whenever $fg \in p$ we have either $f \in p$ or $g \in p$

<u>Lemma</u> (a) If p is prime then p is radical

(b) If p is prime, $p = I \cap J$ then $p = I$ or $p = J$.

<u>Proof</u> (a) If $f^N \in p$ then $f \cdot f^{N-1} \in p \Rightarrow$ either $f \in p$ or $f^{N-1} \in p \Rightarrow$ by induction $f \in p$

(b) Suppose $p \neq I, p \neq J$ find $f \in I, g \in J$ such that $f, g \notin p$. Then $fg \in I \cap J = p$, contradiction.

<u>Def</u> Y = algebraic subset in $\mathbb{A}^n$ is called <u>irreducible</u> if it cannot be written as the union $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are <u>proper</u> alg. subsets of Y.

<u>Thm</u> Y is irreducible $\Leftrightarrow I(Y)$ is prime

<u>Proof</u> 1) Assume $I(Y)$ is prime, $Y = Y_1 \cup Y_2$

Then $I(Y) = I(Y_1) \cap I(Y_2) \Rightarrow$ by Lemma either $I(Y_1) = I(Y)$ or $I(Y_2) = I(Y)$. $\Rightarrow$

either $Y_1 = Z(I(Y_1)) = Z(I(Y)) = Y$ or $Y_2 = Y$.

So Y is irreducible

2) Assume Y is irreducible, let us prove $I(Y)$ is prime. Suppose $fg \in I(Y)$ then $fg = 0$ on Y

1 $\Rightarrow Y = (\{f=0\} \cup \{g=0\})$

Define $Y_1 = Y \cap \{f=0\}$, $Y_2 = Y \cap \{g=0\}$

then $Y_1 \cup Y_2 = Y \overset{Y \text{ irred}}{\Rightarrow}$ either $Y_1 = Y$ or $Y_2 = Y$

$\Rightarrow$ either $f \in I(Y)$ or $g \in I(Y)$. So $I(Y)$ is prime

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Thm Assume $\mathbb{K}$ is alg. closed. Then $\mathbb{A}^n$ is irreducible.

Proof $\mathbb{A}^n = Z(0)$, so by Nullstellensatz

$I(\mathbb{A}^n) = \sqrt{0} = 0$. Since 0 is a prime ideal in $\mathbb{K}[x_1 \dots x_n]$

(prove it!), $\mathbb{A}^n$ is irreducible.

Noetherian rings

Def R = ring is called Noetherian if any ascending (increasing) chain of ideals $I_1 \subset I_2 \subset I_3 \subset \dots$ eventually stabilizes, that is, there exist N such that $I_k = I_N$ for $k \geq N$.

Lemma R is Noetherian $\Leftrightarrow$ any ideal I is finitely generated

Proof: (1) Assume R is Noetherian, I = ideal

Define a sequence of polynomials

$$f_1 \in I, \quad f_2 \in I \setminus (f_1), \quad f_3 \in I \setminus (f_1, f_2), \dots$$

If $(f_1 \dots f_k) = I$, we stop, otherwise find new f_{k+1}.

If we do not stop get

$$(f_1) \subsetneq (f_1, f_2) \subsetneq (f_1, f_2, f_3) \subsetneq \dots$$

② Assume any ideal is finitely generated, let us prove R is Noetherian: $I_1 \subset I_2 \subset I_3 \subset \dots$

Define $I = \bigcup_{k \geq 1} I_k$, let us prove it is an ideal

- $f \in I$, g any $\Rightarrow$ $f \in I_k$ for some k $\Rightarrow$ $fg \in I_k \subset I$.
- $f_1 \in I_k$, $f_2 \in I_\ell$ $\Rightarrow$ $f_1, f_2 \in I_{max(k,\ell)}$ $\Rightarrow$ $f_1 + f_2 \in I_{max(k,\ell)} \subset I$

By assumption, $I = (f_1 \dots f_m)$ and

$f_1 \in I_{k_1}, \dots f_m \in I_{k_m}$

$\Rightarrow$ all $f_i \in I_{max(k_1 \dots k_m)}$ $\Rightarrow$ $I = (f_1 \dots f_m) \subset I_{max(k_1 \dots}$

$\Rightarrow$ $I = I_{max}$ and we are done.

□

<u>Thm</u> If R is Noetherian then $R[x]$ is Noetherian.

<u>Pf</u> $I \subset R[x]$ ideal, let us prove it is finitely generated

$I \ni f = a_n x^n + \dots + a_0$ $\quad a_i \in R$, $a_n \neq 0$

Define $a_{top} = a_n$ and $deg(f) = n$.

Define a sequence of polynomials in I:

- f_1 = element of smallest degree in I
- f_2 = elem of smallest degree in $I \setminus (f_1)$

⋮

- f_k = element of smallest degree in $I \setminus (f_1 \dots f_{k-1})$.

If $(f_1 \dots f_k) = I$, we stop and I is finitely generate.

Otherwise consider a sequence of ideals

$$(a_{top}(f_1), \dots a_{top}(f_k)) \subset R$$

Since R is noetherian, at some point

$$a_{top}(f_m) \in (a_{top}(f_1), \dots, a_{top}(f_k)) \text{ for } m \geq k$$

write $a_{top}(f_m) = \sum_i a_{top}(f_i) u_i$ then

for $g = f_m - \sum u_i x^{\deg(f_m) - \deg(f_i)} f_i$

top coeff cancels and $\deg g < \deg f_m$

$\Rightarrow$ by our assumpt $g \in (f_1 \dots f_{m-1}) \Rightarrow$

$$f_m \in (f_1 \dots f_{m-1}).$$

<u>Corollary</u> (Hilbert Basis Theorem) $\mathbb{K}$ = field

$\mathbb{K}[x_1, \dots x_n]$ is Noetherian $\Rightarrow$ any ideal in $\mathbb{K}[x_1 \dots x_n]$ is finitely generated.

<u>Corollary</u> $Z_1 \supset Z_2 \supset Z_3 \supset \dots$ <u>descending</u> chain of algebraic subsets $\Rightarrow$ Z_i stabilize.

<u>Proof</u> $I(Z_1) \subset I(Z_2) \subset \dots$ $\Rightarrow$ by Noetherian property $I(Z_k)$ stabilize $\Rightarrow$ Z_k stabilize.

<u>Thm</u> Z = algebraic set, then we can write

$$\underbrace{Z = Z_1 \cup \dots \cup Z_k}_{\text{finitely many}} \quad \text{where } Z_i \text{ are irreducible.}$$

<u>Pf</u> If Z is irreducible, we stop. Otherwise $Z = Z_1 \cup Z_2$.

If these are irreducible, we are done, otherwise $Z_1 = Z_1' \cup Z_1''$ and so on.

$$\begin{tikzcd}[row sep=small] & & Z_1' \\ & Z_1 \arrow[ur] \arrow[dr] & \\ Z \arrow[ur] \arrow[dr] & & Z_1'' \quad \dots \\ & Z_2 \arrow[r] & Z_2' \end{tikzcd}$$

Each chain of proper subsets eventually stop $\Rightarrow$ tree is