② Assume any ideal is finitely generated, let us prove R is Noetherian: $I_1 \subset I_2 \subset I_3 \subset \dots$

Define $I = \bigcup_{k\geq 1} I_k$, let us prove it is an ideal

- $f \in I$, g any $\Rightarrow f \in I_k$ for some $k \Rightarrow fg \in I_k \subset I$.
- $f_1 \in I_k$, $f_2 \in I_\ell \Rightarrow f_1, f_2 \in I_{max(k,\ell)} \Rightarrow f_1 + f_2 \in I_{max(k,\ell)} \subset I$

By assumption, $I = (f_1, \dots f_m)$ and

$f_1 \in I_{k_1}, \quad \dots \quad f_m \in I_{k_m}$

$\Rightarrow$ all $f_i \in I_{max(k_1, \dots k_m)} \Rightarrow I = (f_1 \dots f_m) \subset I_{max(k_1 \dots k_m)}$

$\Rightarrow I = I_{max}$ and we are done. □

Thm If R is Noetherian then $R[x]$ is Noetherian.

Pf $I \subset R[x]$ ideal, let us prove it is finitely generated

$I \ni f = a_n x^n + \dots + a_0 \qquad a_i \in R, \; a_n \neq 0$

Define $a_{top} = a_n$ and $deg(f) = n$.

Define a sequence of polynomials in I:

- f_1 = element of smallest degree in I
- f_2 = elem of smallest degree in $I \setminus (f_1)$

⋮

- f_k = element of smallest degree in $I \setminus (f_1 \dots f_{k-1})$.

If $(f_1 \dots f_k) = I$, we stop and I is finitely generate.

Otherwise consider a sequence of ideals

$$(a_{top}(f_1), \dots a_{top}(f_k)) \subset R$$

Since R is noetherian, at some point

$a_{top}(f_m) \in (a_{top}(f_1), \dots, a_{top}(f_k))$ for $m \geq k$

write $a_{top}(f_m) = \sum_i a_{top}(f_i) u_i$ then

for $g = f_m - \sum u_i x^{\deg(f_m) - \deg(f_i)} f_i$

top coef cancels and $\deg g < \deg f_m$

$\Rightarrow$ by our assumpt. $g \in (f_1, \dots, f_{m-1}) \Rightarrow$

$f_m \in (f_1, \dots, f_{m-1})$. ⊗

<u>Corollary</u> (Hilbert Basis Theorem) $\mathbb{K}$ = field

$\mathbb{K}[x_1, \dots, x_n]$ is Noetherian $\Rightarrow$ any ideal in $\mathbb{K}[x_1, \dots, x_n]$ is finitely generated.

<u>Corollary</u> $Z_1 \supset Z_2 \supset Z_3 \supset \dots$ <u>descending</u> chain of algebraic subsets $\Rightarrow$ Z_i stabilize.

<u>Proof</u> $I(Z_1) \subset I(Z_2) \subset \dots \Rightarrow$ by Noetherian property $I(Z_k)$ stabilize $\Rightarrow Z_k$ stabilize.

<u>Thus</u> Z = algebraic set, then we can write

$$Z = \underbrace{Z_1 \cup \dots \cup Z_k}_{\text{finitely many}}$$ where Z_i are irreducible.

<u>Pf</u> If Z is irreducible, we stop. Otherwise $Z = Z_1 \cup Z_2$.

If these are irreducible, we are done, otherwise $Z_1 = Z_1' \cup Z_1''$ and so on.

$$\begin{tikzcd}[row sep=small] & & Z_1' \\ & Z_1 \arrow[ur, dash] \arrow[dr, dash] & \\ Z \arrow[ur, dash] \arrow[dr, dash] & & Z_1'' \quad \dots \\ & Z_2 \arrow[r] \arrow[dr] & Z_2' \\ & & Z_2'' \end{tikzcd}$$

Each chain of proper subsets eventually stop $\Rightarrow$ tree is finite and all leaves are irred.

Lecture 6 | **Lemma 1** Z = algebraic set $= A \cup B$ ← alg. set

$Y \subset Z$ irreducible alg. set

Then $Y \subset A$ or $Y \subset B$.

Proof: $Y = (Y \cap A) \cup (Y \cap B)$

Since Y is irreducible, either $Y = Y \cap A$ or $Y = Y \cap B$

so $Y \subset A$ or $Y \subset B$.

Lemma 2 $Z = Z_1 \cup \dots \cup Z_k$, Z_i irreducible

Assume $Z = A \cup B$ where A, B are alg. set.

Then A = (union of some Z_i)

B = (union of some Z_i).

Proof By lemma 1, either $Z_i \subset A$ or $Z_i \subset B$. So

Since $\bigcup_{i=1}^{k} Z_i = Z$, we have $A = \bigcup_i Z_i : Z_i \subset A$.

can skip.

Def Z_i is called an irreducible component of Z

if (1) $Z_i \subset Z$, Z_i irreducible

(2) It is maximal: if $Z_i \subset Z' \subset Z$ and Z' irreducible then $Z_i = Z'$.

Thus $Z = Z_1 \cup \dots \cup Z_k$ where Z_i are irreducible components of Z. Furthermore, such Z_i are uniquely determined by Z.

Proof (Existence) Recall that $Z = Z_1 \cup \dots Z_k$ where Z_i are irreducible.

Assume for some i Z_i is not a component then $\exists Z'$: $Z_i \subsetneq Z' \subset Z$ and Z' irred.

By Lemma 1, $Z' \subset Z_j$ for some j, so $Z_i \subset Z_j$ and we do not actually need Z_i:

$$Z = \bigcup_{l \neq i} Z_l$$

~~Now we proceed by ind~~ Now we proceed by induction untill we get smallest number of Z_i which must be irreducible components.

(Uniqueness) Assume $Z = Z_1 \cup \_ \cup Z_k = Y_1 \cup \_\_ \cup Y_l$ where Y_i, Z_j are irred. components. Then by Lemma $Y_i \subset Z_j$ for some j $\Rightarrow$ since Y_i are irred. componen[ts] $Y_i = Z_j$ and the result follows.