Lecture 7 | Zariski topology

We define topology on $\mathbb{A}^n$

closed subsets = algebraic sets

open subsets = complements of algebraic sets.

<u>Lemma</u> This is a topological space

① $\emptyset$ closed $\{1=0\}$ $\longleftrightarrow$ $\mathbb{A}^n$ open

② $\mathbb{A}^n$ closed $\{0=0\}$ $\longleftrightarrow$ $\emptyset$ open

③ Z_α closed $\Rightarrow \bigcap_\alpha Z_\alpha$ closed $\longleftrightarrow$ U_α open, $U_\alpha = \mathbb{A}^n \setminus Z_\alpha$
arbitrary intersections $\Rightarrow \bigcup_\alpha U_\alpha$ open arbitrary unions.

④ Z_1, Z_2 closed $\Rightarrow Z_1 \cup Z_2$ closed $\longleftrightarrow$ U_1, U_2 open finite
finite union $U_1 \cap U_2$ open intersection.

<u>Note</u>: Over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$

$\{$open in Zariski topology$\} \Longrightarrow \{$open in usual topology$\}$.

Indeed, polynomials are continuous, so $\{f=0\}$ is closed in usual topology.

<u>The converse is not true!</u>

<u>Ex</u>: in $\mathbb{A}^1$, the only closed subsets are $\emptyset$, $\mathbb{A}^1$ or finite sets.

Lemma 1) Any two nonempty open subsets in $\mathbb{A}^n$ intersect

2) Any open nonempty subset in $\mathbb{A}^n$ is dense.

Proof: 1) Assume $U_1 = \mathbb{A}^n \setminus Z_1$, $U_2 = \mathbb{A}^n \setminus Z_2$

$U_1 \cap U_2 = \emptyset \iff Z_1 \cup Z_2 = \mathbb{A}^n$

But $\mathbb{A}^n$ is irreducible $\Rightarrow Z_1 = \mathbb{A}^n$ or $Z_2 = \mathbb{A}^n \Rightarrow U_1 = \emptyset$ or $U_2 = \emptyset$.

2) Assume $U = \mathbb{A}^n \setminus Z$, and $\overline{U} = Z' =$ Zariski closure

Then $Z' \cup Z = \mathbb{A}^n \Rightarrow$ either $Z' = \mathbb{A}^n$ (so U is dense)

or $Z = \mathbb{A}^n$ (so $U = \emptyset$). $\square$

Cor Zariski topology is not Hausdorff.

! **Warning** Zariski topology on $\mathbb{A}^2$ is not the product topology for Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

Still, we have the following:

Lemma a) If $Z_1 \subset \mathbb{A}^n$ and $Z_2 \subset \mathbb{A}^m$ are closed then $Z_1 \times Z_2$ is closed in $\mathbb{A}^{m+n}$

b) If $U_1 \subset \mathbb{A}^n$, $U_2 \subset \mathbb{A}^m$ are open then $U_1 \times U_2$ is open in $\mathbb{A}^{m+n}$

c) the diagonal $\Delta \subset \mathbb{A}^n \times \mathbb{A}^n$ is closed.

Pf a) Z_1 closed $\Rightarrow Z_1 \times \mathbb{A}^m$ closed (same equations)

$Z_1 \times Z_2 = (Z_1 \times \mathbb{A}^m) \cap (\mathbb{A}^n \times Z_2)$ closed

b) $U_1 = \mathbb{A}^n \setminus Z_1$, $U_2 = \mathbb{A}^n \setminus Z_2$

$\mathbb{A}^{m+n} \setminus (U_1 \times U_2) = (Z_1 \times \mathbb{A}^m) \cup (\mathbb{A}^n \times Z_2)$ closed

c) $\Delta = \{x_1 = x_{n+1}, x_2 = x_{n+2}, \dots, x_n = x_{2n}\}$ closed in $\mathbb{A}^{2n}$

Def $X \subset \mathbb{A}^n$ any subset. We define Zariski topology on X as the induced topology:

$\{\text{closed}\} = X \cap Z$, $Z \subset \mathbb{A}^n$ closed

$\{\text{open}\} = X \cap U$, $U \subset \mathbb{A}^n$ open.

Lemma If X is algebraic in $\mathbb{A}^n$ then

$$\left\{\begin{array}{l}\text{closed subset}\\ \text{of } X\end{array}\right\} = \left\{\begin{array}{c}\text{alg. subsets}\\ Z \subset X\end{array}\right\}$$

Pf: 1) Assume $Z \subset X$ algebraic, then Z closed in $\mathbb{A}^n$ and $Z = Z \cap X \Rightarrow$ closed in X

2) $Z \subset X$ closed in induced topology $\Rightarrow Z = X \cap Z'$ algebraic

$\Rightarrow Z$ is algebraic.

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Lemma X = irreducible top. space, $U \subset X$ open, $U \neq \emptyset$.

Then U is dense in X and irreducible.

Proof: 1) $\overline{U}$ = closure of U = intersection of all closed subsets containing U

Then $X = (X \setminus U) \cup \overline{U}$ closed

~~But~~ X is irreducible and $X - U \neq X \Rightarrow \underline{\overline{U} = X}$.

(2) Suppose $U = Z_1' \cup Z_2'$ where Z_i' closed in U.
Then $Z_i' = Z_i \cap U$ for some closed $Z_i \subset X$
and $Z_1 \cup Z_2 \supset U$. Since $Z_1 \cup Z_2$ is closed and
$\overline{U} = X$ we get $Z_1 \cup Z_2 = X$. Since X is irreducible,
we get $\cancel{Z_1} = X$ or $Z_2 = X \Rightarrow Z_1' = U$ or $Z_2' = U$.

Cor $\mathbb{A}^{n+1} \setminus \{0\}$ is irreducible.

Def Zariski topology on $\mathbb{P}^n$

$\{\text{closed sets}\} = \{\text{projective algebraic sets}\} = Z(I)$

I = homogeneous ideal in $\mathbb{K}[x_0, \ldots, x_n]$

Lemma $\mathbb{P}^n$ is irreducible

Pf: $\mathbb{P}^n = Z_1 \cup Z_2$, consider projection $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$
Since $Z_j = Z(I_j)$ I_j homogeneous, then $\pi^{-1}(Z_i)$ is closed
in $\mathbb{A}^{n+1} \setminus \{0\}$, $\pi^{-1}(Z_1) \cup \pi^{-1}(Z_2) = \mathbb{A}^{n+1} \setminus \{0\}$.
But $\mathbb{A}^{n+1} \setminus \{0\}$ is irreducible, and the result follows.