

MAT 248A, Winter 2024

Homework 2

Due before 3pm on Wednesday, January 21

Please write the homework solutions in connected sentences and explain your work. Mark the answers to each question. Scan or take pictures of your homework and upload it to Gradescope before due time.

In all problems we work over an arbitrary field \mathbf{k} unless stated otherwise.

1. Let I and J be two ideals in $\mathbf{k}[x_1, \dots, x_n]$.

- Assume $f \in I \cap J$, prove that $f^2 \in I \cdot J$.
- Use (a) to prove inclusions $I \cdot J \subset I \cap J \subset \sqrt{I \cdot J}$.
- Prove $\sqrt{I \cdot J} = \sqrt{I \cap J}$.

2. Let $I = (y, y - x^2)$.

- Find $Z(I)$.
- Find \sqrt{I} .

3. a) Denote by $N(n, d)$ the dimension of the space of degree d homogeneous polynomials in n variables. Compute $N(n, d)$.

b) Suppose that P_1, \dots, P_k are points in \mathbb{P}^n and $k < N(n + 1, d)$. Prove that there exists a (nonzero) degree d homogeneous polynomial $f(x_0, \dots, x_n)$ such that

$$f(P_1) = \dots = f(P_k) = 0.$$

4.. A **quadric** is a subset $\{f = 0\} \subset \mathbb{P}^n$ where f is a nonzero homogeneous polynomial of degree 2.

a) Given 5 points in \mathbb{P}^2 , prove that there is a quadric containing all these points.

Hint: Use problem 3.

b) Given 9 points in \mathbb{P}^3 , prove that there is a quadric containing all these points.

Hint: Use problem 3.

c) Given 3 lines in \mathbb{P}^3 , prove that there is a quadric containing all these lines.

Hint: choose 3 points on each line and use (b).