Nonstandard Analysis and Applications

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Presentation notes
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Abstract

Leibniz and Newton, both independently credited as inventors of calculus, relied on the concept of an infinitesimal (nonzero "numbers" that were "infinitely small") in their development. Our standard rigorous treatment of calculus involves an "arbitrary epsilon" limit definiton. There's an alternative rigorous study of calculus beyond the limits of real analysis. In 1961, Robinson constructed the "hyperreal line" as a direct consequence of the compactness theorem of first order logic. We will examine some typical proofs of known statements in advanced calculus and extend the nonstandard framework to other mathematical fields.

1 Introduction to First-Order Logic

1.1 First-Order Languages

Define a first-order language to be a set of symbols, as a base containing a symbol for the logical NAND, quantifiers $\exists$ and $\forall$, equality ($=$), grouping parenthesies/brackets, and variables (as many as are needed). Though NAND is all that is needed, for sanity's sake, one usually includes the usual set of logical connective symbols used for propositional logic formulas: $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$.

In addition to all of these, a specific language is specified by the addition of any number of n-ary function symbols and n-ary relation symbols. S is the set defined to contain as members these extra symbols, and we denote by L^S the set of valid first-order sentences that can be made from S along with the base symbols. (Without going into detail about what is or is not a "well-formed formula", for the purposes of our talk, we can just suppose that we can read off the formula without ambiguity, and it "reads like" a normal mathematical senctence.)

Let's take the example of group theory. We need a 0-ary function whose symbol will be e. We also need a 2-ary function symboled by m (for multiplication). So, $S = \{e, m\}$. Then, the group axioms are given (albeit awkwardly) as the three-element set:

$$\Phi = \{\forall x \forall y \; m(mxy)z = mx(myz), \;\; \forall x \; mxe = x, \;\; \forall x \exists y \; mxy = e\} \subset L^S$$

We feel like with this set up, we can do anything, but we don't have the ability to call on subsets.

1.2 The Completeness Theorem

We say $\Phi \subset L^S$ is *consistent* if there is no formula $\varphi \in L^S$ such that φ "follows from" Φ and $\neg\varphi$ also follows from Φ. $\Phi \subset L^S$ is *maximally consistent* if for all $\varphi \in L^S$ one has φ follows from Φ or $\neg\varphi$ follows from Φ.

Theorem 1.1 *(Henkin) Every consistent set Φ can be extended to a maximally consistent set that "contains witnesses".*

Theorem 1.2 *(Gödel) For $\Phi \subseteq L^S$ and $\varphi \in L^S$, if Φ implies φ is true, then there is a proof of φ with Φ as hypothesis.*

Both of the theorems listed are hardly complete, and it is evident in their very wording that the actual statements are much more rigorous. But as a corollary to Gödel's theorem, we have

Corollary 1.3 *Fix a set Φ of consistent first-order sentences. Then a first-order sentence φ, one has φ is true from Φ iff φ can be proved from Φ.*

Since a theorem will only need a finite number of hypothesis statements out of Φ, we can argue the following: Though there is a proof from Φ to φ, the proof only used a finite subset $\Phi_0 \subset \Phi$. So there is a proof of φ from finite set Φ_0. Thus, by the equivalence granted in the corollary, we have the Compactness Theorem of First-Order Logic:

Theorem 1.4 *(Compactness) If φ follows from Φ, then there is a finite $\Phi_0 \subset \Phi$ such that φ follows from Φ_0.*

As for the Completeness theroem itself, the basic idea of the proof is to take L^S and order consistent subsets by set inclusion. If either of the two subsets being compared is inconsistent, there is no comparison. Then, Zorn's Lemma would grant a set of maximal consistency.

2 The construction of ${}^*\mathbb{R}$

3 An alternate construction: filters

4 Nonstandard proofs

4.1 Proofs from analysis

4.2 Statements in other fields

References

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