Quiz 2 Solutions

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\boxed{\text{Method 1}}$

$$n^2 \le n^2+1$$

$$\frac{1}{n^2} \geq \frac{1}{n^2+1}$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^2}$ converges by the p-test (since $p = 2 > 1$).

Since the terms $\frac{1}{n^2+1}$ are all positive, and we have the inequality facing the right way,

$$\sum \frac{1}{n^2} \geq \sum \frac{1}{n^2+1}$$

So, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Since all the terms are positive, the series converges absolutely. (You could also mention the series passes the requirements of the Absolute Convergence Theorem.)

ALTERNATE SOLUTION

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\boxed{\text{Method 2}}$

$f(x) = \frac{1}{x^2+1}$ is the matching function.

So, we'll examine the improper integral $\int_1^{\infty} \frac{1}{x^2+1}\,dx$

Since $\int \frac{1}{x^2+1}\,dx = \arctan x + C$, we get

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,dx &= \left(\lim_{t\to\infty} \arctan t\right) - \arctan 1 \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4}\end{aligned}$$

Since the improper integral $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test.

Since all the terms are positive, the series converges absolutely. (You could also mention the series passes the requirements of the Absolute Convergence Theorem.)

2. $\sum_{n=0}^{\infty} \frac{n!}{e^n}$

We'll try the ratio test. $a_n = \frac{n!}{e^n}$, so $a_{n+1} = \frac{(n+1)!}{e^{n+1}}$

So, the sequence of ratios is:

$$r_n = \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \frac{(n+1)\cdot n!}{e^1 \cdot e^n} \cdot \frac{e^n}{n!} = \frac{n+1}{e}$$

$\rho = \lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{n+1}{e} = \infty$ (aka $r_n \to \infty$, aka <u>sequence</u> r_n diverges)

$\rho = \infty > 1$, so by the Ratio Test, the <u>series</u>

$$\sum_{n=0}^{\infty} \frac{n!}{e^n}$$

diverges.