Chapter 4

Sequences, Series, and The Integral Test

[1]Your work on this project will lay part of the foundation that will be needed to find a new way of looking at functions. This first step focuses on developing a basic understanding of sequences, series, and the convergence of such objects. In addition, you will have a concentrated opportunity for further work on improving your skills at reading, talking, and writing mathematics.

We assume that you will use your text, as necessary, for definitions, theorems, and as a general reference. We will indicate when you should stop and read a section of the text. The exercises in the project, however, are written so that you can learn by discovery – that is, you should master the material covered by completing the project. Your instructor and teaching assistants will be available for help, but they will not do the exercises for you. Remember that, in addition to regular class and discussion sessions, office hours can be a valuable resource for resolving matters that have not been resolved in your discussions with other students.

To complete this project on time, **it is essential that you start working on it at the beginning of the project week**. It is long and you may find that some parts are fairly challenging. Since the normal workload expectation for this course is 2-3 hours outside of class for each hour inside, project completion and discussion of the material should take 12-20 hours. You may find that you'll make the best progress if you work on it for a few hours, take a break to do something else for awhile, and then return to it. Don't be surprised if you get so engrossed in a problem that you think about it while walking to your other classes, brushing your teeth, etcetera. This may seem odd, but it's quite normal.

Finally, whether or not you have studied calculus previously: DON'T SKIP THE PRELIMINARY STUFF on the assumption that you either already "know it all" or don't need to look at the "easy stuff." One of the goals of this course is for you to gain proficiency in writing mathematics, making conjectures, and proving mathematical claims. The exercises that follow are designed to help you do this. Failure to do them would both deprive you of an important part of the course and put you at risk of not knowing the material with the depth of understanding needed for successful completion of exams and/or other assignments.

WHAT TO EXPECT

In this project, we will lead you through initial study of series involving real terms. We'll start with a discussion of sequences of real numbers and then use our understanding of sequences to define series and convergence properties of series. If you look closely, you may see some nice ways to relate our new material to things that you studied in earlier parts of this course.

Once we get past the meaning of things, we are going to want easy ways of checking for information that we need. In the third section, we'll learn about the Integral Test—our first example of methods for checking series for convergence. When we studied methods of integration we learned strategies or rules that helped us to decide on which method to apply. An analog to this approach will unfold as you learn about a collection of convergence tests. We conclude this project with some practice problems and challenges that pull together our ideas.

SHOW YOUR WORK ON THESE PAGES UNLESS OTHERWISE INDICATED. Your completed project should serve as a nice set of notes that can be referred to when studying for exams.

Sequences: Systematic Lists of Reals

When we report a sequence of events that lead up to something, we list the events in the order of occurrence. With this idea in mind, the mathematical meaning of a sequence makes a lot of sense.

Definition *A **sequence** of real numbers is a function, f, from the natural numbers to the reals. We write $f(n) = a_n$ and denote the sequence by*

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\} \qquad \text{or} \qquad \{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

In practice a sequence of reals is given as an infinite list of real numbers. The n^{th} number in the list, $f(n)$ or a_n, is called the **n**

Let's look at some examples.

1. Given $f(n) = \dfrac{3n}{2n^2+1}$, for a natural number n, write out the first four terms of the sequence and write the sequence in "sequence notation."

To get the first term, we start with $n =$ __________. This leads to $a_1 = 1$. Then, $n =$ __________, __________, __________ leads to $a_2 = \frac{6}{9}$, $a_3 = \frac{9}{19}$, and $a_4 =$ ______________. The sequence notation would be

$$\left\{ \rule{4cm}{0.4pt} \right\}_{n=1}^{\infty} \quad \text{or} \quad 1, \frac{6}{9}, \ldots, \frac{3n}{2n^2+1}, \ldots .$$

Check your answers to (1) with the ones given before the Mathstory at the end of this project.

■

As you can see, writing out terms of a sequence when given its generating formula is very straightforward. Going "the other way" presents a greater challenge. The key here is to try to express each term that you are given as the output of a function acting on the number of the term given. Then use what you got to determine a formula for the n^{th} term.

2. Consider the sequence:

$$-4, \frac{6}{2}, \frac{8}{7}, \frac{10}{14}, \frac{12}{23}, \ldots .$$

Let's find a general formula for the n^{th} term of the sequence. We begin by rewriting the sequence so we can see the corresponding values of n:

$$a_1 = -4 \qquad a_2 = \frac{6}{2} \qquad a_3 = \frac{8}{7} \qquad a_4 = \text{—} \qquad a_5 = \text{—}.$$

We notice that each term contains a common factor of 2. Factoring this out yields

$$a_1 = 2(-2) \quad a_2 = \frac{2(3)}{2} \quad a_3 = \frac{2(4)}{\rule{1cm}{0.4pt}} \quad a_4 = \frac{2(5)}{14} \quad a_5 = \rule{2cm}{0.4pt}.$$

With the exception of the first term, it seems that the other factor in each numerator is one larger than the number of the term. If we rewrite the first term so that it, too, is a quotient with a positive numerator, we obtain $a_1 = \frac{2(2)}{\rule{2cm}{0.4pt}}$. This makes it so that the first term also "fits" the observed pattern. This reasoning leads us to a numerator of ________ $(n +$ ________ $)$ for a_n.

As a hint for the denominator, think about perfect squares. Hopefully, you noticed that each denominator is two smaller than a perfect square. Re–writing the terms to show this leads to

$$a_1 = \frac{2(-2)}{\rule{1cm}{0.4pt}} \quad a_2 = \frac{2(3)}{4-2} \quad a_3 = \frac{2(4)}{9-2} \quad a_4 = \frac{\rule{1cm}{0.4pt}}{\rule{1cm}{0.4pt}} \quad a_5 = \frac{2(6)}{\rule{1cm}{0.4pt}}.$$

Comparing each term to the corresponding value of n, we see that $n = 1$ gives us $1 - 2$, while $n = 2$ gives us $2^2 - 2, \cdots, n = 5$ gives us a denominator of $5^2 - 2$. Pulling this together, we claim the denominator $n^2 - 2$ for a_n. Thus, the formula for the sequence is

$$\{a_n\} = \left\{ \frac{2(\underline{\qquad\quad})}{n^2 - \underline{\qquad}} \right\}.$$

3. For each of the following, find a formula for the n^{th} term that “fits” the terms that are given. Answers may not be unique. Try for simple expressions and remember to think in terms of doubling, squaring, taking powers, etc. of natural numbers.

a. $a_1 = 2 \quad a_2 = \frac{9}{4} \quad a_3 = \frac{64}{27} \quad a_4 = \frac{625}{256}$

b. $f(1) = -2,\ f(2) = \frac{4}{5},\ f(3) = \frac{6}{15},\ f(4) = \frac{8}{29},\ f(7) = \frac{14}{95}$

c. $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$

■

Check your answers to (3a), (3b), and (3c) with the ones given before the Mathstory at the end of this project.

Since sequences of reals are functions on the natural numbers, the idea of the limit of a sequence comes directly from our understanding of limits of functions. If we want to know what is happening with our list of terms in the long run, we need to investigate the behavior of the (sequence) function as n goes to infinity. Recall that if we have a function on the nonnegative reals, in MAT 21A (or its equivalent), we learned that

> $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$, there exists an $M = M(\epsilon) > 0$ such that $x > M$ implies that $|f(x) - L| < \epsilon$.

For a sequence, we restrict ourselves to natural numbers for the arguments of the functions. This leads to

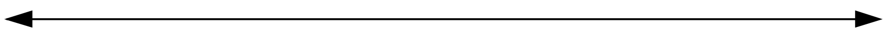
> $\lim_{n\to\infty} f(n) = L$ if and only if for every $\epsilon > 0$, there exists a positive integer N such that $|f(n) - L| < \epsilon$ whenever $n > N$.

4. Translating this to sequence notation, where $f(n) = a_n$, leads to the definition of the

> **Limit of a Sequence**: We say that the limit of a sequence $\{a_n\}$ is L as n approaches infinity, written $\lim_{n\to\infty} a_n = L$, if and only if for every _________, there exists a positive integer $N = N(\epsilon)$ such that
>
> $$|a_n - L| < \epsilon \qquad \text{whenever} \qquad \_\_\_\_\_\_\_\_\_.$$
>
> If a sequence has a limit, then we say it **converges**; otherwise we say it **diverges**. If a_n becomes arbitrarily large as n gets large, then we write $\lim_{n\to\infty} a_n = \infty$.

■

5. It would be nice to have a picture of what this definition says. Suppose that $\lim_{n\to\infty} a_n = L$. Then for $\epsilon > 0$, on the number line below, show the *challenge interval* $|a_n - L| < \epsilon$.



In your own words, describe what it means to say that there exists an $N > 0$ such that

$$|a_n - L| < \epsilon \qquad \text{for all} \qquad n > N$$

in terms of the challenge interval.

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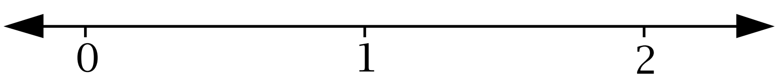
6. On the number line below, show the first five terms of $1/n$.



For each of the following, find a positive integer N such that

a. $n > N$ implies $\dfrac{1}{n} < \dfrac{1}{20}$

b. $n > N$ implies $\dfrac{1}{n} < 0.0001$

Check your answer to (6b) with the ones given before the Mathstory at the end of this project. ■

Problem (6) showed only that we could meet the challenge for two specific ϵ's. That is not good enough to prove $\lim_{n\to\infty} \frac{1}{n} = 0$. We have two ways to show that a given sequence is convergent to a specific number. The first is to offer an $N - \epsilon$ proof, analogous to the $\delta - \epsilon$ proofs we learned in MAT21A. The second is to apply limit theorems. We will practice both ways.

N–ϵ Proofs: A New Look at an Old Friend

Before getting started, we introduce a "rounding up" or "ceiling" function that will be useful in N–ϵ proofs.

For x real, let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x.

7. Note that $\lceil 5 \rceil = 5$, $\lceil 3.2 \rceil = 4$, and $\left\lceil -10\frac{3}{4} \right\rceil = -10$. With this in mind, sketch the graph of $f(x) = \lceil x \rceil$.

■

In general,

for $x > 0$, $\lceil x \rceil$ is always a positive integer and $\lceil x \rceil \geq x$.

8. Look at the first $N - \epsilon$ proof, fill in the four blanks, and make sure that you understand the proof. If you have trouble following the proof, *seek help* from your instructor, TAs for MAT 21C, or friends. Then, fill in what is missing to complete the $N - \epsilon$ proofs for (b) and (c).

a. Use the definition of the limit of a sequence to prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

We want to show that, given any __________, there exists a positive ________________ N such that

$$\left|\frac{n}{n+1} - \underline{\quad\quad}\right| < \epsilon$$

whenever n ______ N.

Let $\epsilon > 0$ be given and suppose that $n > N$, where $N = N(\epsilon)$ is ours to define. Now

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-n-1}{n+1}\right| = \frac{1}{n+1}$$

(we can drop the absolute value bars here because n is a positive integer). But $n+1 > n$ implies that $\frac{1}{n+1} < \frac{1}{n}$ and $\frac{1}{n} < \epsilon$ whenever $n > \frac{1}{\epsilon}$. Choose $N(\epsilon) = \left\lceil \frac{1}{\epsilon} \right\rceil$. Then

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$$

for all $n > N(\epsilon) \geq \left\lceil \frac{1}{\epsilon} \right\rceil$ and we are done.

Check your answers to (8a) with the ones given before the Mathstory at the end of this project.

b. Use the definition to prove that $\lim_{n\to\infty} \frac{n-1}{n^3+1} = 0$.

We want to show that, for ________________ $\epsilon > 0$, there exists a ________________ N such that $\left|\underline{\quad\quad\quad\quad\quad\quad}\right| < \epsilon$ whenever $n > N$.

Let $\epsilon > 0$ be given and suppose that $n > N = N(\epsilon) \geq 1$, where $N(\epsilon)$ is ours to define. Now

$$\left|\frac{n-1}{n^3+1} - 0\right| = \frac{n-1}{n^3+1},$$

where the absolute value signs aren't needed because $n - 1 \geq 0$ for n a natural number. We know that $n - 1 < n$ and $n^3 + 1 > n^3$ implies that $\frac{1}{n^3+1} < \frac{1}{n^3}$. Combining these leads to

$$\left|\frac{n-1}{n^3+1} - 0\right| = \frac{n-1}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2} < \epsilon$$

whenever $n > N(\epsilon) = \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil$. Therefore, given any $\epsilon > 0$, for $n > N(\epsilon) \geq \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil$,

$$\left|\frac{n-1}{n^3+1} - 0\right| < \epsilon$$

and we conclude that $\lim_{n\to\infty} \frac{n-1}{n^3+1} = 0$.

c. Use the definition of the limit of a sequence to prove that $\lim_{n\to\infty} \frac{e^{-n}}{n^2+1} = 0$.

We want to show that, for any __________, ____________________________________

____

____

Let $\epsilon > 0$ be given and suppose that $n >$ ____________________ ≥ 1 where $N(\epsilon)$ is ours to define. Now

$$\left|\frac{e^{-n}}{n^2+1} - 0\right| = \left|\rule{4cm}{0.4pt}\right| = \frac{e^{-n}}{n^2+1}.$$

Because $n > 1$, $e^{-n} < 1$. Hence $\frac{e^{-n}}{n^2+1} <$ ____________________ $< \frac{1}{n^2}$. Substituting yields that

$$\left|\frac{e^{-n}}{n^2+1}\right| < \frac{1}{n^2} < \epsilon$$

whenever $n > N(\epsilon) = \left\lceil \rule{2.5cm}{0.4pt} \right\rceil$. Thus, given any $\epsilon > 0$, for $n >$ __________ $\geq \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil$,

$$\left|\frac{e^{-n}}{n^2+1} - 0\right| < \frac{1}{n^2} < \epsilon$$

and we conclude that ____. ■

9. For a little additional practice, use the definition of the limit of a sequence to prove each of the following.

a. $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$.

b. $\lim_{n\to\infty} \frac{\sin n}{n^2+1} = 0$.

■

The good news is that we are done with using the definition to prove the limits of specific sequences. The brief review was to allow time to revisit a skill on which you spent much energy in MAT 21A and, more importantly, to underscore the extent to which the concept of a limit serves as a foundation for Calculus.

Limit Theorems: Express Lane to Our Goal

From here on, we will find limits of sequences (and later, of series) by making use of the ways for calculating limits that were developed for finding the limits of functions. Recall that

$$\lim_{x\to\infty} \frac{1}{x^k} = 0 \qquad \text{for } k = 1, 2, 3, \ldots \qquad \text{and } x \in \mathbb{R} - \{0\}\,.$$

10. Make use of this and the limit theorems (or any other tools) you know from MAT 21A to compute the following limits, where x is real.

a. $\displaystyle\lim_{x\to\infty} \frac{3x}{5x+8}$

b. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{2x}\right)^x$

c. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

Check your answers to (10) with the ones given before the Mathstory at the end of this project.

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In a sequence, we can make the same sort of statement as was made above, only now it looks like

$\lim_{n\to\infty} \frac{1}{n^k} = 0$ for $k = 1, 2, 3, \cdots$ (n passes through the natural numbers).
Notice here that the base n changes while the exponent k is fixed.

From the theorems we had in MAT 21A about the limits of combinations of functions, we immediately claim:

Limits of Combinations of Sequences.
If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, where A and B are real numbers, then

i. $\lim_{n\to\infty} (a_n + b_n) = A + B$

ii. $\lim_{n\to\infty} (a_n - b_n) = A - B$

iii. $\lim_{n\to\infty} (a_n b_n) = AB$

iv. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$, if $B \neq 0$, and,

v. if k is a constant, $\lim_{n\to\infty} (ka_n) = kA$.

Other useful properties include:

$$\lim_{n\to\infty} \frac{k^n}{n!} = 0 \quad \text{for } k = 1, 2, 3, \ldots \quad (n \text{ passes through the natural numbers})$$

and $n! = n(n-1)(n-2)\ldots(2)(1)$. Notice that, in contrast to the preceding, the base k is fixed while the exponent n changes.

and

$$\lim_{n\to\infty} r^n = 0 \qquad \text{when } |r| < 1.$$

Notice here that n is the exponent while r is fixed.

Now read pp. 575–580 of our text and check your skill level with the next problem.

11. Determine whether or not the given sequences converge.

a. $\{a_n\} = \left\{\frac{5n}{7n+12}\right\}_{n=2}^{\infty}$

b. $a_n = \left\{ \left(1 - \frac{3}{n}\right)^n \right\}_{n=1}^{\infty}$

Check your answer to (11b) with the ones given before the Mathstory at the end of this project.

■

If you don't feel comfortable with your skill level, try problems 5, 7, 9 and 11 on page 583 of our text.

It's a nice time for a break!

Digging Deeper: More Convergence Tests

For more involved sequences, there are three other useful tools. All of these are based on old friends from MAT 21A. In that setting, they were defined for real-valued functions. Here, we reintroduce them for functions on the positive integers.

The **first tool** is the

Squeeze Principle: Let $f(n)$, $g(n)$, and $h(n)$, where n is a positive integer, be sequences. If

$$g(n) \leq f(n) \leq h(n) \quad \text{and} \quad \lim_{n\to\infty} g(n) = L = \lim_{n\to\infty} h(n),$$

then

$$\lim_{n\to\infty} f(n) = L.$$

The **second tool** is the

Theorem on Bounded Monotone Sequences: If the sequence of reals $\{a_n\}$ is nondecreasing and bounded above, then $\{a_n\}$ is convergent; if $\{a_n\}$ is nonincreasing and bounded below, then it is convergent.

Fill in what is missing in the following restatement of the Theorem on Bounded Monotone Sequences. Suppose $\{u_n\}$ is an infinite sequence of reals.

If ...	then ...
$u_n \le u_{n+1}$ for each n and there exists a real number B such that $u_n \le B$ for all n	$\{u_n\}$ converges.
$u_n \ge u_{n+1}$ for each n and there exists a real number A such that ______ $\le$ ______ for all n	$\{u_n\}$ converges.

12. We don't have enough mathematical background to prove the Theorem on Bounded Monotone Sequences. It is a good idea, though, to think about its plausibility. Let's look at the nondecreasing case. Fill in what is missing in order to make the following discussion complete.

Let $\{u_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers such that

(i) $u_n \le u_{n+1}$ for each natural number n and

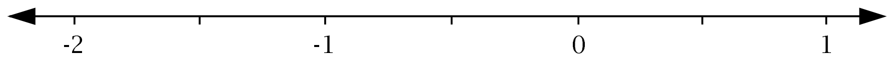
(ii) there exists a real number B such that $u_n \le B$ for all n.

a. Suppose that you were told that B is not the "best upper bound" for u_n. Write a sentence or two describing what that would mean to you.

b. Now suppose that B is the best upper bound; for example, the best upper bound for $\left\{1 - \dfrac{2}{n}\right\}$ is 1, while the best upper bound for $\left\{\left(1 + \dfrac{1}{n}\right)^n\right\}$ is ______ .

Both of the examples in (b) show that the best upper bound **doesn't need to be a member of the sequence**. However, you do need to be able to **get as close to it as you would like** by going out far enough in the sequence. Moreover, you must **never get past it**.

c. For example, given the sequence $\left\{1 - \dfrac{1}{2n}\right\}_{n=1}^{\infty}$, show 7 terms on the number line below.



Now find a natural number N such that, for $n > N$, $\left(1 - \frac{1}{2n}\right) > \frac{3}{4}$.

Find a natural number N such that, for $n > N$, $\left(1 - \frac{1}{2n}\right) > \frac{7}{8}$.

d. In general, because $-\frac{1}{2n} <$ ______, we see that $1 + \left(-\frac{1}{2n}\right) < 1$. For $\epsilon > 0$, choose $N = N(\epsilon) \geq \left\lceil \frac{1}{2\epsilon} \right\rceil$. Then $\frac{1}{2n} <$ ______ whenever $n > N(\epsilon) \geq \left\lceil \frac{1}{2\epsilon} \right\rceil$ and $1 - \frac{1}{2n} >$ ____________ whenever $n > N(\epsilon)$.

e. Pulling this together: when a sequence $\{a_n\}$ is nondecreasing, once we "get past" a challenge number c with the k^{th} term; i.e., $a_k > c$, we know that $a_n > c$ for all $n > k$. If B is the best upper bound, then we always have $a_n \leq B$. If we call the challenge number $B - \epsilon$, then for any $\epsilon > 0$, we have that there exists a positive integer N such that for all n, $a_n \geq$ ______, i.e.,

$$B - \epsilon \leq a_n \leq B$$

for all $n \geq N$.

f. **If we could prove** that there were a best upper bound for a nondecreasing sequence that is bounded above, we would know that the best upper bound is the ____________ of the sequence. Note that the existence of a best upper bound can be proved using Advanced Calculus (brief commercial message: this is done in MAT 127).

■

Before describing the third tool, let's practice using the Theorem on Bounded Monotone Sequences.

13. Use the theorem on Bounded Monotone Sequences to justify that each of the following sequences is convergent. We'll lead you through the first one.

a. $\left\{\frac{2}{n}+\left(\frac{1}{3}\right)^n\right\}_{n=1}^{\infty}$

First, to use the Theorem on Bounded Monotone Sequences, we have to show that the sequence either is nondecreasing and bounded above, or is nonincreasing and bounded below.

For this sequence, we note that $\frac{2}{n} \geq 0$ and $\left(\frac{1}{3}\right)^n \geq 0$ for each n. Therefore, the sequence $\left\{\frac{2}{n}+\left(\frac{1}{3}\right)^n\right\}_{n=1}^{\infty}$ is bounded ____________ by 0.

Now we have to show that it is non____________________. Let $u_n = \frac{2}{n}+\left(\frac{1}{3}\right)^n$. Then $u_{n+1} \leq u_n$, that is, u_n is nonincreasing, if and only if ____________________ ≥ 0. But

$$\begin{aligned} u_n - u_{n+1} &= \left(\frac{2}{n}+\left(\frac{1}{3}\right)^n\right) - \left(\rule{4cm}{0.4pt}\right) \\ &= \left(\frac{2}{n}-\frac{2}{n+1}\right) + \left(\rule{4cm}{0.4pt}\right) \\ &= \frac{2}{n(n+1)} + \left(\frac{1}{3}\right)^n \left(\rule{4cm}{0.4pt}\right) \\ &> 0. \end{aligned}$$

Therefore, u_n is non______________. By the Theorem on ______________ ______________ ______________, $\{u_n\}$ is ________________.

Check your answers to (13a) with the ones given before the Mathstory at the end of this project.

b. $\left\{\ln\frac{n}{n+1}\right\}_{n=1}^{\infty}$

(Note: to show that $\ln\frac{n}{n+1}$ is increasing, you can work with the function $f(x) = \ln\frac{x}{x+1}$

for $x > 0$ and use a fact about increasing functions that you learned in MAT 21A.)

■

The **third useful tool** is

l'Hôpital's Rule.

However, to apply it, **be careful**: recall that a function can be differentiated only if it is continuous, whereas a sequence is defined as a function from the positive integers to the reals. Thus, we cannot apply l'Hôpital's Rule directly to the sequence. Instead,

from the given function $a_n = f(n)$, define a function $h(x)$ on the reals that becomes the function f when x is replaced by n. Use l'Hôpital's Rule on the function $h(x)$ over the reals. The conclusion applies to the function $f(n)$.

We demonstrate what we mean by this in the following problem.

14. Consider the sequence $\left\{ \frac{\ln n}{n} \right\}$. To claim convergence, we want to find the

$$\lim_{n\to\infty} \frac{\ln n}{n}.$$

To use l'Hôpital's Rule, define a continuous function

$$f(x) = \frac{\ln x}{x} \text{ for } x > 0.$$

By l'Hôpital's Rule, we have that $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} =$ ______, where x is real. A subset of the values of f, namely, those corresponding to positive integer values of x, must exhibit the same behavior. Hence, we conclude that

$$\lim_{n\to\infty} \frac{\ln n}{n} = \underline{\hspace{2cm}}.$$

Therefore, the sequence ________________ to __________. ■

15. Determine the convergence or divergence of the following sequences. Remember that you can use the limit theorems, the tools referred to above, or any other mathematically correct arguments that build on what you learned about the limit of functions of x as x goes to ∞.

a. $\left\{\dfrac{\sin(3n)}{n-7}\right\}_{n=9}^{\infty}$

b. $\left\{\dfrac{e^{n+4}}{n^2}\right\}_{n=1}^{\infty}$

c. $\left\{\dfrac{3^n}{4^n-50}\right\}_{n=3}^{\infty}$

d. $\{a_n\} = \left\{\sqrt{n^2+n} - 2n\right\}$

e. Do problem 10, from p. 583, of our text. Do the work in the space provided below.

Check your answers to (15c) and (15d) with the ones given before the Mathstory at the end of this project. ■

Take a break if you haven't done so recently.

Series

The reason for our long discussion of sequences is because they are used both to define a series and to define convergence of a series. This is a very important and useful mathematical tool.

When we get hooked on a TV series (like 'The Young and the Restless,' 'Deep Space Nine,' 'Dr. Quinn: Medicine Woman,' or 'ER'), we hate to miss an episode because the episodes build on each other or add to previous information. With this idea in mind, the mathematical meaning of a series makes a lot of sense.

Given an infinite sequence $\{a_k\}_{k=1}^{\infty}$, define a new sequence $\{S_n\}_{n=1}^{\infty}$, called the **sequence of n$^{\text{th}}$ partial sums**, by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

The associated **infinite series** is given by

$$\sum_{k=1}^{\infty} a_k.$$

The series $\sum_{k=1}^{\infty} a_k$ is said to **converge** if and only if the sequence $\{S_n\}$ converges. Furthermore, if $\lim_{n\to\infty} S_n = S$ we write $\sum_{k=1}^{\infty} a_k = S$ and call S the sum of the series. The series $\sum_{k=1}^{\infty} a_k$ diverges if and only if the sequence $\{S_n\}_{n=1}^{\infty}$ **diverges**.

Because convergence of series was defined in terms of the convergence of a particular sequence, some of our limit properties for sequences carry over to series. For example,

Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent to the real number A, and $\sum_{n=1}^{\infty} b_n$ is convergent to the real number B. Then

(i) if c is a constant, $\sum_{n=1}^{\infty} ca_n$ is convergent to cA,

(ii) $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent to the real number $A + B$, and

(iii) $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent to the real number $A - B$.

Be careful. The preceding only works if *both* series converge. Also note that we are not making claims about the series $\sum_{n=1}^{\infty} (a_n b_n)$ or $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)$.

The following is a useful property to keep in mind when manipulating series. Namely, we have that

Adding or deleting a finite number of terms does not affect the convergence or divergence of an infinite series. Thus, we can ignore the first few (or ten, or thousand) terms of a series and claim convergence or divergence according to what can be shown for the rest.

Exploring the Definition

We will appeal to the definition of convergence (divergence) of series in order to develop a repertoire of approaches for checking for convergence, not as the standard tool to investigate series. The examples given in the next problem are meant to build our comfort and familiarity with a process.

16. Use the definition to decide on the convergence or divergence of each of the following series.

a. Use the definition to prove that the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is convergent.

Using partial fraction decomposition (or by taking a flying leap and checking our guess), we note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{(k+1)}.$$

We will use this to find the sequence of n^{th} partial sums. Hence,

$$\begin{aligned}
S_1 &= \sum_{k=1}^{1} \frac{1}{k(k+1)} = 1 - \tfrac{1}{2} \\
S_2 &= \sum_{k=1}^{2} \left(\frac{1}{k} - \frac{1}{(k+1)}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = \underline{\qquad\qquad} - \underline{\qquad\qquad} \\
S_5 &= \sum_{k=1}^{5} \left(\frac{1}{k} - \frac{1}{(k+1)}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\underline{\qquad\qquad}\right) + \left(\underline{\qquad\qquad}\right) + \left(\underline{\qquad\qquad}\right) + \left(\underline{\qquad\qquad}\right) \\
&= 1 - \underline{\qquad\qquad\qquad}
\end{aligned}$$

This is an example of what is called a **telescoping sum** (you may have seen these in MAT 21B). In general, we see that

$$S_n = \sum_{k=1}^{\overline{}} \frac{1}{k} - \frac{1}{(k+1)} = 1 - \underline{\qquad\qquad\qquad}.$$

Thus, $\lim_{n\to\infty} S_n =$ ____________ and we conclude that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} =$ ____________.

b. Show that $\sum_{k=1}^{\infty} \frac{2}{4k^2-1}$ is a telescoping sum and use the definition to show that the series is convergent.

Note that $\frac{2}{4k^2-1} = \frac{1}{2k-1} -$ ____________
This leads to

$S_1 =$

$S_2 =$

$S_3 =$

In general, it looks like

$S_k =$

which indicates that $\sum_{k=1}^{\infty} \frac{2}{4k^2-1}$ is a ____________ sum. Furthermore, since

$$\lim_{k\to\infty} S_k = \_\_\_\_\_\_\_\_\_\_\_\_,$$

we can conclude that $\sum_{n=1}^{\infty} \frac{2}{4k^2-1}$ is ____________ to ________.

Take a moment to write a few sentences describing what is happening in these telescoping sums.

c. Use the definition to prove that the series $\sum_{k=1}^{\infty} \sin\frac{\pi k}{2}$ is divergent.

We need to look at the sequence of n^{th} partial sums:

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{1} \sin\frac{\pi n}{2} = \sin\frac{\pi}{2} = 1 \\
S_2 &= \sum_{k=1}^{2} \sin\frac{\pi n}{2} = \sin\frac{\pi}{2} + \sin\pi = \underline{\qquad\qquad} \\
S_3 &= \sum_{k=1}^{3} \sin\frac{\pi n}{2} = \overbrace{\sin\frac{\pi}{2} + \sin\pi} + \sin\frac{3\pi}{2} = \underline{\qquad\qquad} \\
S_4 &= \sum_{k=1}^{4} \sin\frac{\pi n}{2} = 0 + \sin 4\frac{\pi}{2} = \underline{\qquad\qquad} \\
S_5 &= \sum_{k=1}^{5} \sin\frac{\pi n}{2} = 0 + \sin 5\frac{\pi}{2} = \underline{\qquad\qquad} \\
S_6 &= \sum_{k=1}^{6} \sin\frac{\pi n}{2} = \underline{\qquad\qquad} \\
S_7 &= \sum_{k=1}^{7} \sin\frac{\pi n}{2} = \underline{\qquad\qquad} \\
S_8 &= \sum_{k=1}^{8} \sin\frac{\pi n}{2} = \underline{\qquad\qquad}
\end{aligned}
$$

In general,

$$
S_n = \begin{cases} 1, & \text{if } n = 4k-3; \\ 1, & \text{if } n = 4k-2; \\ 0, & \text{if } n = 4k-1; \\ 0, & \text{if } n = 4k, \end{cases}
$$

where k is a natural number. But the sequence

$$1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \ldots$$

is not $\underline{\qquad\qquad\qquad}$. Therefore, the series $\sum_{n=1}^{\infty} \sin\frac{\pi n}{2}$ is divergent.

■

The Geometric Series or Bust

In Discussion Set A2.1 from MAT 21A, an “argument” about “arrow travel” was offered to make the claim that William Tell’s arrow wouldn’t ever make it to the apple on his son’s head. To refresh your memory, we said

Suppose that William Tell was standing L feet away from his son when he released the arrow. We will consider the flight of the arrow, taking increments of time based on how far the arrow has traveled. After time t_1, the arrow has gone half the distance, or $L/2$ feet. Then the second time measurement, t_2, is the time it takes to go half the remaining distance, or $L/4$. Continuing in this fashion, after time $t_1 + t_2 + t_3 + t_4 + t_5$, the arrow will have gone

$$\frac{L}{2} + \frac{L}{4} + \frac{L}{8} + \frac{L}{16} + \frac{L}{32} = \frac{31}{32}L.$$

So, the arrow hasn't quite gotten to the apple yet. But we can continue this process indefinitely. Since the arrow only goes half the remaining distance to the apple at each interval, the apple is never reached.

Situations like this–called paradoxes–were raised by philosophers, such as Zeno of Elea (c. 495 B.C.E.), long ago to illustrate issues. Look up the word "paradox" in your dictionary and write the definition that you find here.

A paradox is

Zeno's paradoxes were based on the assumption that either time and space could be infinitely divided **or** they were made up of indivisible elements. The Greeks called the indivisible elements "atoms;" today, mathematicians refer to them as infinitesimals.

The most widely accepted response of mathematicians to philosophical paradoxes such as the one above involve series and derivatives. The one we look at now leads us to "the best known" series. We'll illustrate the mathematician's response to Zeno's paradox known as The Dichotomy.

17. **The Dichotomy**: A runner in a 100 meter race never crosses the finish line.

Zeno's argument: The runner must reach the first halfway-to-the-finish point (the 50 meter mark). The runner must reach the second halfway-to-the-finish point (the 75 meter mark). Then the runner must reach the third halfway-to-the-finish point (the __________ meter mark). Continue this indefinitely; there are an infinite number of half-way-to-the-finish points and there is a positive distance between any two of them. Since the distance between halfway-to-the-finish points is positive, it takes a positive amount of time to get from one to the next. So, our runner never reaches the finish line because the time required to finish is the sum of infinitely many positive numbers!

A Mathematician's Response: Before we begin, let's look at the sequence $\left\{\left(\frac{1}{10}\right)^n\right\}_{n=1}^{\infty}$.

a. What is $\left(\frac{1}{10}\right)^n$ for $n = 1, 2, 3, 7$?

b. What happens as $n \to \infty$? (We saw this earlier, in a more general form.)

c. Simplify each of the following:

1. $(1-x)\left(1+x+x^2\right) =$

2. $(1-x)\left(1+x+x^2+x^3+x^4+x^5\right) =$

3. $(1-x)\left(1+x+x^2+\ldots+x^n\right) =$

d. If $S = 1+x+x^2+\ldots+x^n$, the formula you just obtained tells us that

$$S = \frac{\underline{\hspace{3cm}}}{\underline{\hspace{3cm}}}.$$

Therefore,

$$\begin{aligned}
1+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)^2+\ldots+\left(\frac{1}{10}\right)^5 &= \underline{\hspace{3cm}} \\
1+(1.1)+(1.1)^2+\ldots+(1.1)^{200} &= \underline{\hspace{3cm}} \\
\tfrac{1}{2}+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^3+\ldots+\left(\tfrac{1}{2}\right)^{401} &= \underline{\hspace{3cm}}
\end{aligned}$$

Check your answers to (17d) with the ones given before the Mathstory at the end of this project.

e. Now let's go back to the Dichotomy.

As of the first halfway point, the runner has gone $H_1 = \frac{1}{2}(100)$ meters.

As of the second halfway point, the runner has gone $H_2 = \frac{1}{2}(100) + \frac{1}{2}\left(\frac{1}{2}(100)\right)$ meters.

As of the third halfway point, the runner has gone $H_3 = \frac{1}{2}(100) +$ __________ $+$ __________ meters. Factoring the 100 gives $H_3 = \left(\underline{\qquad\qquad}\right)(100)$ meters.

Write the sum that represents how far the runner has gone as of the 10^{th} halfway point:

$$H_{10} = (\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad})(100) \text{ meters.}$$

Now look at the sum. In general, for the n^{th} halfway point,

$$H_n = (\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad})(100) \text{ meters.}$$

Using our "short formula" from (d) for this sum,

$$\begin{aligned} H_n &= \frac{1}{2}(\underline{\qquad\qquad\qquad\qquad})(100) \text{ meters} \\ &= \frac{1}{2}\frac{\underline{\qquad\qquad}}{\left(1 - \frac{1}{2}\right)}(100) \text{ meters} \\ &= \left(\underline{\qquad\qquad}\right)(100) \text{ meters.} \end{aligned}$$

As $n \to \infty$, $H_n \to$ __________.

f. At the finish line, there is a lovely red ribbon that hangs into the 100 meter distance by 1/100 of a meter. Find an n that gets the runner past the point of touching the ribbon.

Check your answers to (17f) with the ones given before the Mathstory at the end of this project.

■

The mathematician's response lead us to looking at the series

$$\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots$$

which can be rewritten as

$$\frac{1}{2}\left(1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\cdots\right) \text{ or } \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}.$$

This is called a geometric series with common ratio $\frac{1}{2}$. It is our first example of a type of series that we want to recognize "at first sight" because the convergence or divergence properties are known.

A **geometric series** is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots \qquad \text{(where } a\neq 0\text{)}.$$

Notice that a geometric series is characterized by the fact that the ratio of any term to the preceding term is always the same number r. A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to the sum $\frac{a}{1-r}$ if $-1<r<1$ and diverges otherwise.

The convergence (divergence) claim for geometric series can be seen from our work in (17d): The n^{th} partial sum of $\sum_{k=1}^{\infty} ar^{k-1}$ is

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1} = a\left(1+r+r^2+\ldots+r^{n-1}\right) = a\frac{1-r^n}{\rule{2em}{0.4pt}}.$$

What happens to the partial sums as $n\to\infty$, when $-1<r<1$?

Geometric series are great: If we spot one, the common ratio instantly tells us whether or not the series converges; if the common ratio, r, is between -1 and 1, we can use the constant multiple, a, and the common ratio to find the sum. Remember that, to use our formula, we want to "factor out" what is needed so that the remaining series starts with 1. For example, given

$$\sum_{k=4}^{\infty} \frac{7 \cdot 3^k}{5^k} = 7\left(\frac{3}{5}\right)^4 + 7\left(\frac{3}{5}\right)^5 + 7\left(\frac{3}{5}\right)^6 + \cdots$$

we have that

$$\sum_{k=4}^{\infty} \frac{7 \cdot 3^k}{5^k} = 7\left(\frac{3}{5}\right)^4 \left[1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right].$$

In the notation above, $a =$ ________ and $r =$ ________.

18. For each of the following, find the common ratio, r, and the constant multiple, a. If $-1 < r < 1$, find the sum.

a. $\displaystyle\sum_{k=6}^{\infty} \frac{2^k 3^{k+1}}{5^{k+2}}$

b. $\displaystyle\sum_{k=3}^{\infty} \frac{3^{k-4} \cdot 5}{2^{k+1}}$

19. Consider the following:

$$\sum_{n=1}^{\infty} (-1)^n \frac{5(2^n)}{3^{n+1}}$$

a. Write out the first five terms of this series.

b. Find the sum of the series.

Check your answer to (19b) with the ones given before the Mathstory at the end of this project.

■

20. We can use geometric series to change repeating nonterminating decimals to fractions. We see that if

$$.\bar{1} = .1111\cdots = \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots$$

and

$$\begin{aligned} .\overline{02} = .020202\cdots &= \frac{2}{10^2} + \frac{2}{10^4} + \frac{2}{10^6} + \cdots \\ &= \frac{2}{10^2} + \frac{2}{\left(10^2\right)^2} + \frac{2}{\left(10^2\right)^3} + \cdots \end{aligned}$$

then

$$.\overline{123} = \underline{\hspace{10cm}}.$$

Now use the formula for the sum of a geometric series to write $.\bar{1}$, $.\overline{02}$, and $.\overline{123}$ as the ratio of two integers.

Check your answers to (20) with the ones given before the Mathstory at the end of this project.

Expanding our Repertoire of Easy Tests

Just as we offered tools for looking at the convergence (or divergence) of sequences, we want a repertoire of strategies that we call Tests for Convergence (or Divergence) for series. We have just seen that recognition of a given series as geometric almost instantly leads to convergence (divergence) information. Our next observation tells us a second thing for which to check. This one, though, either tells us about divergence or tells us nothing.

Divergence "At First Sight"

In this section we want to focus on the behavior of the n^{th} term in a series.

Given $\sum_{k=1}^{\infty} a_k$, we have that $S_n = \sum_{k=1}^{n} a_k$ and $S_{n-1} = \sum_{k=1}^{\overline{\quad}}$____________ for $n > 1$. It follows that

$$S_n - S_{n-1} = \underline{\qquad\qquad}.$$

Now, if $\sum_{k=1}^{\infty} a_k = L$, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} =$ ____________. From this, we conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \underline{\qquad\qquad}$$

which leads us to the following

Fact: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Be careful: the converse is not true. That is, the fact makes no claims for what happens if $\lim_{n\to\infty} a_n = 0$. We will see from examples that $\lim_{n\to\infty} a_n = 0$ does not tell us whether or not $\sum_{n=1}^{\infty} a_n$ converges. That's the bad news. The good news is that we do know from the fact above that if the n^{th} term of a series does not go to zero, then we are sure that the series diverges. Thus we have a test for divergence.

n^th Term Test for Divergence: Consider the infinite series

$$\sum_{k=1}^{\infty} a_k.$$

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

21. Determine which series can be shown to diverge by the n^{th} Term Test for Divergence and which need further study to determine convergence or divergence.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+5}{4n+7}$

b. $\displaystyle\sum_{n=4}^{\infty} \frac{3^n}{2^n+3^n}$

c. $\displaystyle\sum_{n=1}^{\infty} (1-\frac{1}{2n})^n$

d. $\displaystyle\sum_{n=2}^{\infty} \frac{\sin n}{n}$

e. $\displaystyle\sum_{n=1}^{\infty} \cos(3/n)$

f. $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{4n^3+5}$

Check your answers to (21) with the ones given before the Mathstory at the end of this project.

■

Next we want to make sure that you believe that $\lim_{n\to\infty} a_n = 0$ doesn't allow us to claim information about $\sum_{n=1}^{\infty} a_n$. On the way to doing this, we meet a series that we want to recognize "on sight" because it is an important divergent series.

The **harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

is **divergent**.

22. In this problem, we explore the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots.$$

a. Write out S_{16} for this series.

b. Notice that the first term in this partial sum is greater than $1/2$. Put a curly brace, $\underbrace{\quad}$, under this term.

c. Notice that the second term in this partial sum is equal to $1/2$. Put a curly brace, $\underbrace{\quad}$, under this term.

d. Now let's look at the third and fourth terms. Since

$$\frac{1}{3} > \frac{1}{4}, \frac{1}{3}+\frac{1}{4} > \underline{\qquad} + \underline{\qquad} = \frac{1}{2}.$$

Therefore, the sum of the third and fourth terms is greater than $1/2$. Put a single curly brace under these two terms.

e. To keep getting to sums greater than $1/2$, we need to pull together more terms. Let's look at the next four terms. Since $\frac{1}{5} > \frac{1}{8}$, $\frac{1}{6} > \underline{\qquad}$, and $\frac{1}{7} > \frac{1}{8}$, we have that $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad} = \underline{\qquad}$. Therefore the sum of these four terms is greater than ______. Put a single curly brace under these four terms.

f. Now play the same game with the next eight terms. We want to use the fact that each of these is at least as large as ________. Again this gives a sum that is greater than ________. Put a single curly brace under these eight terms.

g. Look over your work for (b) - (f). In each part, we grouped terms to get a sum at least as large as 1/2. We can continue this grouping for selected S_n, and let $n \to \infty$: the sum of the next 16 terms will exceed 1/2, the sum of the 32 terms after that will exceed 1/2, and so on. We get $S_1 = 1$, $S_2 = 1 + \frac{1}{2}$, $S_4 > 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right)$, $S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right)$, $S_{16} >$ ____________________. In general, $S_{2^n} > 1 +$ ______ $\left(\frac{1}{2}\right)$. Since $\lim_{n\to\infty} S_{2^n} =$ ______, the limit does not exist. Therefore the harmonic series diverges.

■

The problem above and (16a) yields an additional lesson. Recall that, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} (a_n + b_n)$ merely is the sum of the two series. Also, recall that we said that this does not work if one or both series diverge. Go back and look over your work in (16a): we proved that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges. However, note that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{(k+1)}$, and that $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{(k+1)}$ are harmonic series that diverge. The "moral of the story " is that if you are tempted to split a series into component "pieces," **be careful** and make sure that the series involving the "pieces" converge. Better yet, **don't split the series**.

23. Work problems 11, 14, and 22 from page 590 of our text. Do the work in the space

provided below.

■

We end this section underscoring our recent discovery; even though $\lim_{n\to\infty} \frac{1}{n} = 0$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This justifies the claim made before the statement of the n^{th} Term Test. Given a sequence $\{a_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} a_n = 0$ does not tell us whether or not $\sum_{n=1}^{\infty} a_n$ converges. We need more tests!

Break time.

The Integral Test

The divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ tells us that $1/n$ doesn't get small enough fast enough. Since $\frac{1}{n^2} < \frac{1}{n}$ for $n > 1$, it seems like a good idea to investigate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

24. Sketch the graphs of $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$ for $x > 0$ on the axes below.

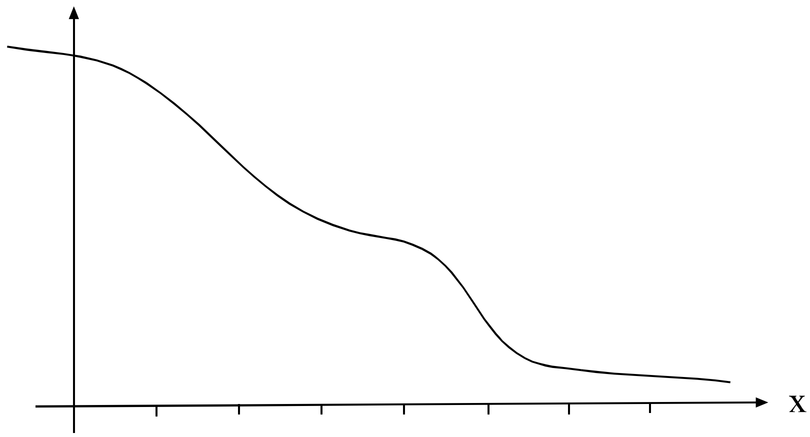
■

Both $\frac{1}{x}$ and $\frac{1}{x^2}$ go to zero, but $\frac{1}{x^2}$ "goes there faster." Looking back at $\frac{1}{x^2}$ when x is a natural number gets us a function on the positive integers, $f(n) = \frac{1}{n^2}$: this is the general term that is added in the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the n^{th} term of this series goes to zero, there is, at least, a chance that the series converges. We're going to obtain that conclusion after we justify a series test known as the Integral Test.

You may be wondering, why integrals? Since a series involves a sum and integrals were defined using sums, it seems like a good idea to try to use integrals to study series. We will see that this is effective for some — but not all — series.

To relate an integral to the series $\sum_{n=1}^{\infty} f(n)$, we need a function that is defined for real x, $x \geq 1$, and that relates easily back to the series. If we consider $f(x)$ created by replacing n with x in the summand, we'll get the logical candidate. (Recall that we also did this when using l'Hôpital's Rule to test convergence of a sequence.) Because we are integrating, we want f *continuous*. Because we want to use estimating sums, we'll soon see that we also want f *positive* and *decreasing* for $x \geq 1$.

25. We consider here a generic continuous, positive function that is decreasing for $x \geq 1$. Its graph is shown below.

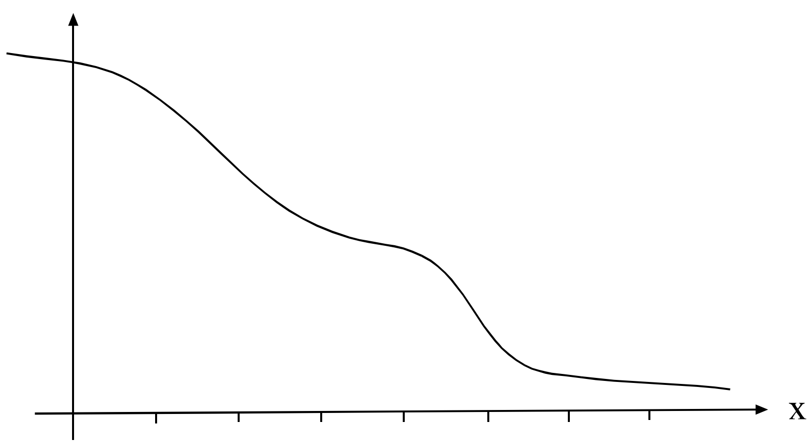


a. On the graph, use the partition $\{1, 2, 3, 4, 5, 6\}$ to draw rectangles corresponding to the upper estimating sum; i.e., an over estimate, for the area. Shade the rectangles.

b. Since part (a) was done for a partition of $[1, 6]$ having 5 sections of equal length, let U_5 denote the sum that gives the area you shaded in (a). Write out a formula for U_5.

$$U_5 = f(1) \cdot 1 + \_\_\_\_\_\_ + \_\_\_\_\_\_ + f(4) \cdot 1 + \_\_\_\_\_\_$$

c. The graph below is a copy of the graph from (a). Use the partition $\{1, 2, 3, 4, 5, 6\}$ to draw rectangles corresponding to the lower estimating sum; i.e., an under estimate, for the area. Shade the rectangles.



d. Let L_5 denote the sum that gives the area you shaded in (c). Write out a formula for L_5.

$$L_5 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

e. From our understanding of the definite integral, we know that

$$\_\_\_\_\_\_\_\_ \le \int_1^6 f(x)dx \le U_5.$$

f. Find the n^{th} partial sum for $\sum_{k=1}^{\infty} f(k)$ when $n = 5$ and when $n = 6$.

$S_5 =$ ________________________

$S_6 =$ ________________________

g. Comparing what we obtained in (b) and (d) with what we got in (f) leads to

$$S_5 = \_\_\_\_\_\_\_\_ \qquad \text{and} \qquad S_6 = L_5 + \_\_\_\_\_\_\_\_.$$

h. Combining the result from (g) with that from (e), we obtain

$$\int_1^6 f(x)\,dx \le U_5 = \_\_\_\_\_\_\_\_$$

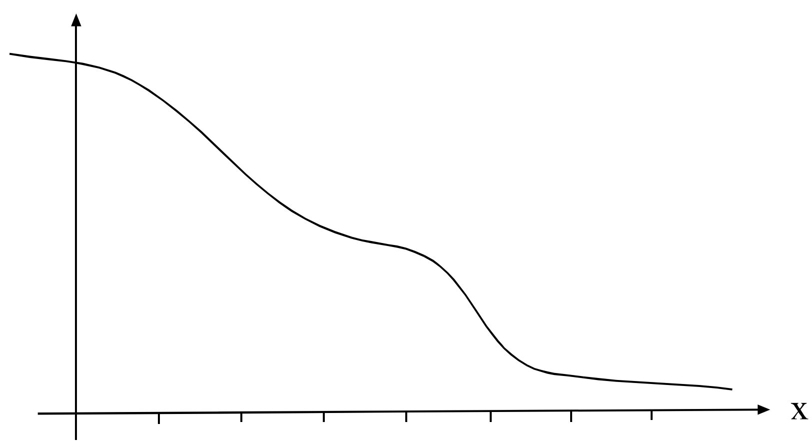
and

$$S_6 - f(1) = L_5 \le \int_1^{\_\_} \_\_\_\_\_\_\_\_.$$

■

26. There is nothing magical about the endpoint 6 in problem (25). Let's quickly fill in what we get if we change it to $n+1$ for an arbitrary n. Our partition now is $\{1, 2, 3, \ldots, n, n+1\}$ which has n segments each having length 1.

a. The graph of our generic function is shown below.



Now the upper estimating sum $U_n =$ ________________________________;
the lower estimating sum $L_n =$ ________________________________;
and we have

$$L_n \leq \int_1^{n+1} f(x)\,dx \leq U_n. \tag{4.1}$$

b. We also have that

$$S_n = \sum_{k=1}^{n} \_\_\_\_\_\_\_\_\_\_ \qquad \text{and} \qquad S_{n+1} = \sum_{k=\_\_}^{\_\_} \_\_\_\_\_\_\_\_\_\_.$$

Comparing these to our estimating sums gives

$$S_n = \_\_\_\_\_\_\_\_\_\_ \qquad \text{and} \qquad S_{n+1} = L_n + f(1).$$

c. This, with (4.1) leads us to two inequalities that relate

$$\int_1^{n+1} f(x)\,dx$$

to a partial sum. We have

$$\int_1^{n+1} f(x)\,dx \leq \_\_\_\_\_\_\_\_\_\_ \tag{4.2}$$

and

$$S_{n+1} - f(1) \leq \int_1^{n+1} f(x)\,dx. \tag{4.3}$$

d. Look at (4.2) and discuss (with your classmates, your instructor, your TA) what conclusion you could draw if $\int_1^{\infty} f(x)\,dx$ is a divergent improper integral. Write a brief statement about your conclusion.

e. Look at (4.3) and the Theorem on Bounded Monotone Sequences and discuss (with your classmates, you instructor, you TA) what conclusion you could draw if $\int_1^\infty f(x)\, dx$ is a convergent improper integral. Write a brief statement about your conclusion.

■

The work you did in (26) gives a plausibility argument for the

Integral Test: Let $f(x)$ be a continuous, positive, decreasing function for $x \geq 1$.

$$\text{If} \quad \int_1^\infty f(x)\,dx \text{ converges, then} \quad \sum_{n=1}^{\infty} f(n) \text{ converges.}$$

$$\text{If} \quad \int_1^\infty f(x)\,dx \text{ diverges, then} \quad \sum_{n=1}^{\infty} f(n) \text{ diverges.}$$

27. In using the integral test, it is important to check (and state) that $f(x)$ is ________________, continuous, and ________________. In some cases, the lower limit of the integral, and hence the index of the sum, must be adjusted to ensure this. This is legitimate because we can ignore a finite number of terms in an infinite series without affecting its convergence or divergence.

Read pp. 592–593 through Example 1, in our text now.

Using the Integral Test

28. Let's go back to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The function $f(x) = \frac{1}{x^2}$ is ________________, decreasing, and ________________ in $[1, \infty)$, so the Integral Test applies. Since

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \underline{\hspace{4cm}} = \underline{\hspace{2cm}}$$

we conclude that the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ is ________________. Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is ________ by the Integral Test.

■

29. Apply the Integral Test to study the **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a positive real number. **Be careful**: your conclusions may depend on p.

■

If your work for (29) was correct, you should have deduced the

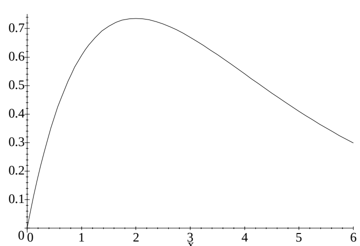
> **Convergence Behavior of the p–series**: The p series
>
> $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
>
> converges for $p > 1$ and diverges for $p \leq 1$.

Caution: Be careful not to confuse p series with geometric series. In a p series, the exponent is fixed and the base changes. In a geometric series, the ratio is fixed and is raised to powers that change.

30. Use the integral test to determine convergence or divergence of the following series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4+n)^{3/2}}$

b. $\displaystyle\sum_{n=1}^{\infty} ne^{-n/2}$; the graph of $xe^{-x/2}$ is shown below.



Check your answer to (30b) with the ones given before the Mathstory at the end of this project.

An Integral to Estimate Error

The work we did in quest of the integral test also leads us to bounds on the n^{th} partial sums and on the sum of a series. However, **the value of a convergent integral does not give us the actual sum of the series**. That is, given the series

$$\sum_{k=1}^{\infty} f(k),$$

let

$$S_n = \sum_{k=1}^{n} f(k) \qquad \text{and} \qquad R_n = \sum_{k=n+1}^{\infty} f(k).$$

If f is a continuous decreasing function that is positive on $[1, \infty)$, with $\int_1^\infty f(x)\,dx$ a convergent improper integral, then

$$\int_1^{n+1} f(x)\,dx \le S_n \le f(1) + \int_1^n f(x)\,dx$$

and

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx.$$

The quantity R_n provides us with a bound on the error in using a convergent improper integral to approximate a sum.

Read pp. 594–596 in our text now.

31. Find bounds on the error incurred if we use $S_6 = \sum_{n=1}^{6} \frac{1}{n^3}$ in a problem where $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is needed.

■

Putting Things Together, Thus Far

At this point, we have two series tests: the n^{th} term test for divergence and the integral test. In addition, we know the convergence/divergence behavior of p–series, geometric series, and harmonic series (a special case of p–series). This is enough for us to start practicing the following process or approach to testing series.

> To classify the series $\sum_{n=1}^{\infty} a_n$:
>
> **(1)** Check to see if $\lim_{n\to\infty} a_n = 0$.
>
> - If $\lim_{n\to\infty} a_n \neq 0$, then state that the series diverges by the n^{th} Term Divergence Test. You are done!
> - If $\lim_{n\to\infty} a_n = 0$, the test fails; i.e., no conclusion can be drawn! Proceed to the next step.
>
> **(2)** Look to see if you recognize the summand.
>
> - If the series is geometric and convergent, find the sum.
> - If the series is a p–series, announce the convergence or divergence by indicating the size of p.
> - If not recognized, proceed to the next step.
>
> **(3)** Apply a series test according to specified conditions.
>
> - If $f(x)$ obtained by replacing n with x in a_n gives a continuous, positive function that is decreasing in $x \geq \ell$ for some positive integer ℓ, then the Integral Test, applied to $\int_{\ell}^{\infty} f(x)dx$, may be useful.
> - If $f(x)$ is not easy to integrate or does not satisfy the hypotheses for the Integral Test, appeal to another test. More tests will be forthcoming.

32. Use the process described above to discuss the convergence of each of the following.

a. $\sum_{n=1}^{\infty} \frac{2^n}{3^{n+5}}$

b. $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

c. $\sum_{j=1}^{\infty} \frac{j+1}{\ln(j+1)}$

■

33. Prove that

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

■

34. From p. 590 of our text, do problem 27.

■

35. From p. 591 of our text, do problem 28.

■

36. In (20) we saw how to write repeating decimals as rational numbers. Do (a) and (b) for more practice and use the experience gained to do (c).

a. Find a rational representation for $0.\overline{21} = 0.21212121\cdots$.

b. Find a rational representation for $0.10135135\overline{135}\cdots$.

c. Show that every repeating decimal represents a rational number.

■

37. Let $a_n = \dfrac{100^n}{n!}$.

a. Show that $a_1 < a_2 < \ldots < a_{99}$.

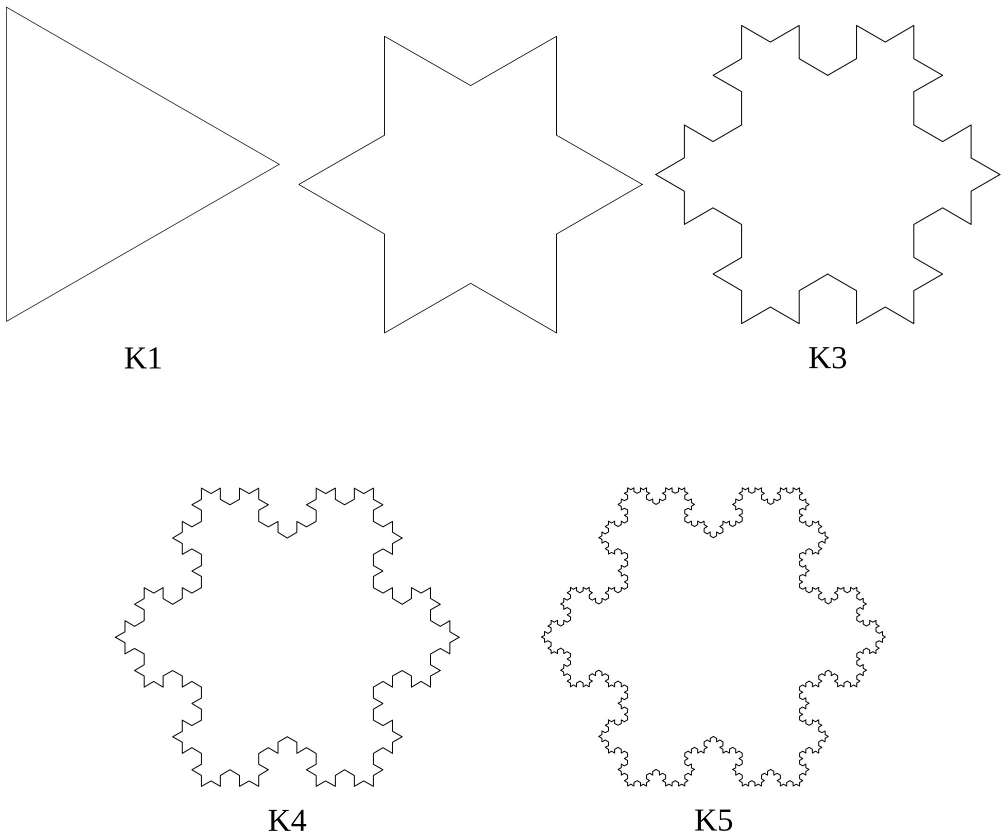
b. Show that $a_{99} = a_{100}$.

c. Show that $a_{100} > a_{101} > a_{102} > \ldots$.

d. Speculate on the convergence of $\{a_n\}_{n=1}^{\infty}$, using some of the material in this project to justify your claims.

■

38. A **fractal** is a figure that is self-similar – that is, it looks similar at any scale of measurement. The **Koch snowflake** is an example of a fractal. It is constructed by beginning with an equilateral triangle, and then iteratively superimposing an equilateral triangle on the middle third of every line segment. The first five iterations of a Koch snowflake are shown below.



Let the length of a side of the Koch snowflake K_1 be L.

a. What is the length of the perimeter of the 1^{st} iterate of the Koch snowflake, K_1?

b. What is the length of the perimeter of the 2^{nd} iterate of the Koch snowflake, K_2?

c. What is the length of the perimeter of the 3^{rd} iterate of the Koch snowflake, K_3?

d. What is the length of the perimeter of the 5^{th} iterate of the Koch snowflake, K_5?

e. Use your work in (a)–(d) to derive a formula for the length of the perimeter of the n^{th} iterate of the Koch snowflake, K_n. Remember to justify your claims and to follow the exposition guidelines given in the "Mathematical Writing" handout.

■

Congratulations! You are now ready to go forth and add to your repertoire of tests for convergence of series. Remember that the process described on page 41 should always be followed while part (3) will expand as you learn more convergence tests.

Mathstories

One of the most brilliant mathematicians of the $20th$ century was **Srinivasa Ramanujan** (1887-1920). Ramanujan, an accounting clerk in India, had little formal mathematical training. However, in 1913, he sent a letter describing some of his results to the prominent

English mathematician G. H. Hardy. Hardy realized that the work of a remarkable genius was in his hands. He arranged for Ramanujan to go to Cambridge. For the next three years, Ramanujan wrote prolifically on his extraordinary discoveries, including many on infinite series. Unfortunately, Ramanujan became ill, due perhaps to a combination of English weather and the limited quantity of vegetable protein available to him. He never completely recovered and died in 1920 at the age of 32.

One of Ramanujan's results was the discovery of an infinite series that can be used to approximate π to a high level of accuracy with very few terms:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

The first four terms of this series gives an approximation for π accurate to 30 decimal places (Edwards and Penney, 1994).

G. H. Hardy (1877-1947) was an analyst — a mathematician who works in the branches of mathematics most similar to what you have learned this year in calculus — and probably the best known English mathematician of his generation. His biography, "A Mathematician's Apology," contains many philosophical, as well as mathematical, gems (his title inspired part of our problem on Zeno's paradox).

> "I believe that mathematical reality lies outside us, that our function is to discover or observe it and the theorems that we describe grandiloquently as our 'creations' are simply the notes of our observations."

Answers to Selected Problems

1. 1; 2, 3, and 4, $12/33$, $3n/(2n^2+1)$

3. (a) $a_n = \left(\dfrac{n+1}{n}\right)^n$, (b) $\dfrac{2n}{2n^2-3}$, (c) $\dfrac{1}{n(n+1)}$;

6b. $N = 10000$ works, as does anything larger

8. (a) $\epsilon > 0$; integer; 1; $>$.

10. $3/5$; $\sqrt{e}$, 0

11. (b) convergent to $1/e^3$

13. (a) below; increasing, $u_n - u_{n+1}$; $\frac{2}{n+1} + \left(\frac{1}{3}\right)^{n+1}$; $\left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^{n+1}$, $1 - \left(\frac{1}{3}\right)$; increasing; Bounded Monotone Sequences; convergent.

15. (c) 0, (d) $-\infty$;

17. (d) $\frac{1-x^{n+1}}{1-x}$; $\frac{10^6-1}{(9)(10)^5}$; (10) $(1.1)^{201} - 10$; $1 - \left(\frac{1}{2}\right)^{401}$;(f)$\left[1 - \left(\frac{1}{2}\right)^n\right] 100 > 99.99$ or $(.5)^n < .0001$; For $n = 14$, $(.5)^{14} \approx .000061$

19. (b) From $a = \frac{-10}{9}$ and $r = \frac{-2}{3}$, the sum is $\frac{-2}{3}$.

20. $0.\overline{123} = \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \cdots$; $0.\bar{1} = \frac{1}{9}$; $0.\overline{02} = \frac{2}{99}$; $0.\overline{123} = \frac{123}{999}$

21. a,b,c, and e diverge by the n^{th} Term Test; d and f need further study.

30. (b) Since $f(x) = xe^{-x/2}$ is positive, continuous, and decreasing for $x \geq 2$, we can apply the Integral Test. Using IBP, the improper integral $\int_2^\infty xe^{-x/2}\,dx$ converges to $(8/e)$. We can conclude that $\sum_{n=2}^{\infty} ne^{-n/2}$ converges and, thus $\sum_{n=1}^{\infty} ne^{-n/2}$ converges.

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