COMPRESSIVE INVERSE SCATTERING I. HIGH FREQUENCY SIMO MEASUREMENTS

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Abstract. The inverse scattering problem with point scatterers and the single-input-multiple-output (SIMO) measurements is analyzed by the compressed sensing techniques with and without the Born approximation.

Three main results about (probabilistic) recoverability of sparse target by the L^1-optimization technique called Basis Pursuit (BP) are obtained in the high frequency limit. In the absence of noise, it is shown that BP can recover exactly the target of sparsity up to the dimension of the measurement either with the multi-shot SIMO measurement for the Born scattering or with the single-shot SIMO measurement for the exact scattering. The stability with respect to noisy data is proved for weak or widely separated scatterers under a stronger sparsity constraint.

1. Introduction

A monochromatic wave u propagating in a heterogeneous medium characterized by a variable refractive index $n = \sqrt{1+\nu}$ is governed by the Helmholtz equation

$$\Delta u(\mathbf{r}) + \omega^2(1+\nu(\mathbf{r}))u(\mathbf{r}) = 0 \tag{1}$$

where $\nu \in \mathbb{C}$ describes the medium inhomogeneities. For simplicity, the wave velocity is assumed to be unity and hence the wavenumber ω equals the frequency.

Consider the plane wave incidence

$$u^{\rm i}(\mathbf{r}) = e^{i\omega \mathbf{r}\cdot\mathbf{d}}$$

where $\mathbf{d} \in S^{d-1}, d = 2, 3$, is the incident direction. The scattered field $u^{\rm s} = u - u^{\rm i}$ then satisfies

$$\Delta u^{\rm s} + \omega^2 u^{\rm s} = -\omega^2 \nu u \tag{2}$$

which can be written as the Lippmann-Schwinger equation:

$$u^{\rm s}(\mathbf{r}) \quad = \quad \omega^2 \int_S \nu(\mathbf{r}')\left(u^{\rm i}(\mathbf{r}') + u^{\rm s}(\mathbf{r}')\right) G(\mathbf{r},\mathbf{r}')d\mathbf{r}' \tag{3}$$

where G is the Green function for the operator $-(\Delta + \omega^2)$.

The scattered field has the far-field asymptotic [28]

$$u^{\rm s}(\mathbf{r}) = \frac{e^{i\omega|\mathbf{r}|}}{|\mathbf{r}|^{(d-1)/2}}\left(A(\hat{\mathbf{r}},\mathbf{d}) + \mathcal{O}\left(\frac{1}{|\mathbf{r}|}\right)\right), \quad \hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|, \quad d = 2, 3 \tag{4}$$

where the scattering amplitude A is determined by the formula [34]

$$A(\hat{\mathbf{r}},\mathbf{d}) \quad = \quad \frac{\omega^2}{4\pi}\int d\mathbf{r}'\nu(\mathbf{r}')u(\mathbf{r}')e^{-i\omega\mathbf{r}'\cdot\hat{\mathbf{r}}}. \tag{5}$$

In the inverse scattering theory, the scattering amplitude is the observable data and the main objective then is to reconstruct ν from the knowledge of the scattering amplitude. In this paper, we use the compressed sensing technique called the *Basis Pursuit* to study the inverse scattering problem for a *discrete* medium composed of point scatterers. Note that since u in (5) is part of the unknown, the inverse scattering problem is nonlinear.

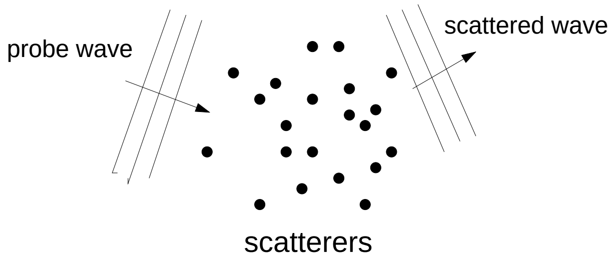

FIGURE 1. Far-field imaging geometry

The standard theory of inverse scattering asserts the injectivity of the mapping from $\nu \in C_c^1$ with a nonnegative imaginary part to the corresponding scattering amplitude for a fixed frequency in *three* (or higher) dimensions (Theorem 5.5 of [9]. See also [19, 28, 32, 33] for similar results, [23, 30, 35] for inverse boundary-value problem and [10, 24, 27, 28] for inverse obstacle scattering). In this case, the refractive index can be determined uniquely by the full knowledge of $A(\hat{\mathbf{r}}, \mathbf{d}), \forall \mathbf{d}, \hat{\mathbf{r}}$, for a fixed ω. Indeed, as A is analytic in both $\mathbf{d}$ and $\hat{\mathbf{r}}$, it suffices to know A for a countably many incident and sampling angles in order to determine ν uniquely. As far as we know, the uniqueness in two dimensions with a fixed frequency is still an open question. What is known for two dimensions is that the uniqueness holds if the scattering amplitude is given for an interval of frequencies [9].

Of obvious theoretical interest, the uniqueness result by itself is of limited practical interest. All existing methods for determining the refractive index without linearizing the problem are based on the methods of constrained nonlinear optimization in the L^2-norm for which the exact recoverability is usually difficult to establish, especially in the case of undersampling [9, 10]. In this paper we show that a target is the unique, global minimizer of an optimization method based on the L^1-norm if the target satisfies certain sparsity constraint. Moreover, this L^1-minimization problem can be effectively solved by linear programming as well as various low-complexity iterative schemes.

In this paper we focus on the two dimensional case with $\mathbf{r} = (x, z) \in \mathbb{R}^2$ noting the relative insensitivity of our approach to the dimension (cf. Remark 2). The three dimensional case is briefly discussed in Section 3.1. Consider the medium with point scatterers located in a square lattice $\mathcal{M} = \{\mathbf{r}_i = (x_i, z_i) : i = 1, ..., m\}$ of spacing ℓ. The total number m of grid points in $\mathcal{M}$ is a perfect square. Without loss of generality, assume $x_j = j_1\ell, z_j = j_2\ell$ where $j = (j_1 - 1)\sqrt{m} + j_2$ and $j_1, j_2 = 1, ..., \sqrt{m}$. Let $\nu_j, j = 1, ..., m$ be the strength of the scatterers. Let $\mathcal{S} = \left\{\mathbf{r}_{i_j} = (x_{i_j}, z_{i_j}) : j = 1, ..., s\right\}$ be the locations of the scatterers. Hence $\nu_j = 0, \forall \mathbf{r}_j \notin \mathcal{S}$. When there is no risk of confusion, we shall write $\nu = (\nu_j)$ in the sequel.

The scattering amplitude for this medium is a finite sum

$$A(\hat{\mathbf{r}}, \mathbf{d}) = \frac{\omega^2}{4\pi} \sum_{j=1}^{m} \nu_j u(\mathbf{r}_j) e^{-i\omega \mathbf{r}_j \cdot \hat{\mathbf{r}}}. \tag{6}$$

Moreover, in analogy to (3), the exciting field $u(\mathbf{r}_{i_j})$ satisfies the Foldy-Lax equation [25, 26, 29]

$$u(\mathbf{r}_{i_l}) = u^{\mathrm{i}}(\mathbf{r}_{i_l}) + \omega^2 \sum_{l \neq j} G(\mathbf{r}_{i_l}, \mathbf{r}_{i_j}) \nu_{i_j} u(\mathbf{r}_{i_j}), \quad l = 1, ..., s \tag{7}$$

where all the multiple scattering effects are included but the self field is excluded to avoid blow-up.

2. METHODS AND RESULTS

2.1. **Born approximation.** First consider the Born approximation to (6). In the Born approximation (also known as Rayleigh-Gans scattering in optics), the exciting field $u(\mathbf{r}_{i_j})$ is replaced by the incident field $u^{\rm i}(\mathbf{r}_{i_j})$. This approximation linearizes the relation between the the target strength and the scattering amplitude and is valid for sufficiently weak or widely separated scatterers.

For the Born scattering, we define the target vector $X = \nu \in \mathbb{C}^m$ and use multiple incident waves

$$u^{\rm i}_k(\mathbf{r}) = e^{i\omega(z\sin\theta_k + x\cos\theta_k)}, \quad k = 1, ..., p \tag{8}$$

where θ_k is the incident angle of the k-th probe wave. Throughout the paper we consider the single-input-multiple-output (SIMO) measurements in which for each incident angle θ_k the resulting scattering amplitude is measured at the multiple sampling angles $\tilde\theta_l, l = 1, ..., n$. After normalization by $\omega^2/(4\pi)$, the totality of the collected data forms the measurement vector $Y \in \mathbb{C}^{pn}$. The corresponding sensing matrix in the linear relationship $Y = \mathbf{A}X$ has the $(n(k-1)+l, j)$-entry

$$e^{-i\omega(z_j\sin\tilde\theta_l + x_j\cos\tilde\theta_l)} u^{\rm i}_k(\mathbf{r}_j). \tag{9}$$

where $\tilde\theta_l$ is the sampling angle of the l-th sensor.

Recent breakthrough in compressed sensing has established the insight that the target can be recovered exactly with nearly minimum sensing resources by the L^1-minimization technique, called Basis Pursuit (BP)

$$\min \|X\|_1 \qquad \text{s.t. } \mathbf{A}X = Y \tag{10}$$

if the target is sufficiently sparse and the matrix $\mathbf{A}$ satisfies either the incoherence property or the restricted isometry property [2, 3, 4, 5, 6, 7]. The L^1-minimization problem (10) can be solved by linear programming techniques [1, 5, 7] or by various greedy algorithms [11, 31, 37].

In this paper we adopt the incoherence approach to analyzing the SIMO inverse scattering problem. Previously the inverse problem for a periodic target with the SIMO measurements has been solved by using the restricted isometry approach [16]. However, the same approach seems unlikely to work here for the case of randomly distributed scatterers.

Let us state the perhaps simplest criterion for exact recoverability of the incoherence approach.

Proposition 1. [12, 18] *BP reconstructs perfectly any target X of sparsity*

$$s \le \frac{1}{2}\left(\frac{1}{\mu(\mathbf{A})} + 1\right) \tag{11}$$

where the sparsity $s = \|X\|_0$ is the number of nonzero components in X and the coherence parameter μ is defined as

$$\mu(\mathbf{A}) = \max_{i\neq j} \frac{\left|\sum_l A_{li}A^*_{lj}\right|}{\sqrt{\sum_l |A_{li}|^2 \sum_l |A_{lj}|^2}}.$$

Proposition 1 implies that the lower the coherence of the sensing matrix is the more massive the exactly recoverable target can be. Under the condition (11), a simple greedy algorithm called Orthogonal Matching Pursuit (OMP) can provably find the minimizer of (10) in at most s iterations [37].

To construct sensing matrices of low coherence let us define the SIMO-sensor ensemble as follows. Let the incident angles $\theta_k, k = 1, ..., p$, be independently and identically distributed according to the probability density function $f^{\rm i} \in C^h([-\pi, \pi])$; for every incident angle let the sampling angles $\tilde\theta_l, l = 1, ..., n$, be independently and identically distributed according to the probability density function $f^{\rm s} \in C^h([-\pi, \pi])$ where $h > 0$ is the degree of smoothness. Define $\text{supp}(f^{\rm i}) = \{\theta : f^{\rm i}(\theta) \neq$

0} and $\mathrm{supp}(f^{\mathrm{s}}) = \{\theta : f^{\mathrm{s}}(\theta) \neq 0\}$. We call $\theta_* \in [-\pi, \pi]$ a *Blind Spot* if there exists a pair $\mathbf{r}, \mathbf{r}' \in \mathcal{M}$ such that

$$\left|(\mathbf{r} - \mathbf{r}') \cdot (\cos\theta_*, \sin\theta_*)\right| = |\mathbf{r} - \mathbf{r}'|. \tag{12}$$

In other words, the set of Blind Spots consists of all the angles between the $x-$axis and $\mathbf{r}-\mathbf{r}', \forall \mathbf{r}, \mathbf{r}' \in \mathcal{M}$.

In Section 3, we prove that the following coherence bound for the sensing matrix with entries (9).

Theorem 1. *Let the sensing matrix* $\mathbf{A}$ *be given according to the sensor ensemble. Suppose*

$$m \leq \frac{\delta}{2} e^{K^2/2}, \quad \delta, K > 0. \tag{13}$$

Then the sensing matrix (9) satisfies the coherence bound

$$\mu(\mathbf{A}) < \left(\chi^{\mathrm{i}} + \frac{\sqrt{2}K}{\sqrt{p}}\right)\left(\chi^{\mathrm{s}} + \frac{\sqrt{2}K}{\sqrt{n}}\right) \tag{14}$$

with probability greater than $(1-\delta)^2$ *where in general* χ^{i} *(resp.* χ^{s}*) satisfies the bound*

$$\chi^{\mathrm{i}} \leq c_t (1 + \omega\ell)^{-1/2} \|f^{\mathrm{i}}\|_{t,\infty}, \tag{15}$$

$$\textit{resp.} \quad \chi^{\mathrm{s}} \leq c_t (1 + \omega\ell)^{-1/2} \|f^{\mathrm{s}}\|_{t,\infty}, \tag{16}$$

where $\|\cdot\|_{t,\infty}$ *is the Hölder norm of order* $t > 1/2$ *and the constant* c_t *depends only on* t. *If, however,* $\mathrm{supp}(f^{\mathrm{i}})$ *(resp.* $\mathrm{supp}(f^{\mathrm{s}})$*) does not contains any Blind Spot, then* χ^{i} *(resp.* χ^{s}*) satisfies the bound*

$$\chi^{\mathrm{i}} \leq c_h (1 + \omega\ell)^{-h} \|f^{\mathrm{i}}\|_{h,\infty}, \tag{17}$$

$$\textit{resp.} \quad \chi^{\mathrm{s}} \leq c_h (1 + \omega\ell)^{-h} \|f^{\mathrm{s}}\|_{h,\infty}, \quad \|f^{\mathrm{i}}\|_{h,\infty} = \sum_{|k| \leq h} \left\| \frac{d^k}{d\theta^k} f^{\mathrm{i}} \right\|_{\infty} \tag{18}$$

where the constant c_h *depends only on* h.

Remark 1. *Theorem 1 along with Proposition 1 then imply that any target of sparsity up to*

$$s \leq \frac{1}{2} + \frac{1}{2}\left(\chi^{\mathrm{i}} + \frac{\sqrt{2}K}{\sqrt{p}}\right)^{-1}\left(\chi^{\mathrm{s}} + \frac{\sqrt{2}K}{\sqrt{n}}\right)^{-1} \tag{19}$$

can be exactly recovered by BP.

If $\omega\ell$ *is sufficiently large (which is the high frequency limit referred to in the title), the dominant term on the right hand side of (19) is*

$$\frac{\sqrt{np}}{4K^2} \tag{20}$$

in view of (15)-(18).

Note that the high frequency limit for the Helmholtz equation is different from that for the Schrödigner equation. The high -frequency quantum scattering is essentially linear (without the Born approximation) and can be solved by the Radon transform [28].

Remark 2. *In three dimensions,* χ^{i} *and* χ^{s} *satisfy different bounds. This is briefly discussed in Section 3.1. For brevity, we state here the result for arbitrary distributions* $f^{\mathrm{i}}, f^{\mathrm{s}} \in C^1$*: Instead of (15)-(16) we have*

$$\chi^{\mathrm{i}} \leq c(1 + \omega\ell)^{-1} \|f^{\mathrm{i}}\|_{1,\infty} \tag{21}$$

$$\textit{resp.} \quad \chi^{\mathrm{s}} \leq c(1 + \omega\ell)^{-1} \|f^{\mathrm{s}}\|_{1,\infty}. \tag{22}$$

Consequently, the asymptotic behavior (20) sets in faster in three dimensions than in two dimension in general.

To improve the sparsity constraint (11), Tropp [39] develops an approach in which the recoverability is only probabilistic in the following ensemble of targets. Let the *target ensemble* consist of target vectors with at most s non-zero entries whose phases are independently uniformly distributed in $[0, 2\pi]$ and whose support indices are independently and randomly selected from the index set $\{1, 2, ..., m\}$.

The following theorem is a reformulation of results due to Tropp [39]. We refer the reader to [17] for the derivation of Proposition 2.

Proposition 2. *Let X be drawn from the target ensemble. Assume that*

$$\mu^2 s \leq \left(8 \ln \frac{m}{\varepsilon}\right)^{-1}, \quad \varepsilon \in (0,1) \tag{23}$$

and that for $q \geq 1$

$$3\left(\frac{q \ln s}{2 \ln \frac{m}{\varepsilon}}\right)^{1/2} + \frac{s}{m}\|\mathbf{A}\|_2^2 \leq \frac{1}{4e^{1/4}}. \tag{24}$$

Then X is the unique solution of BP with probability $1 - 2\varepsilon - s^{-q}$. Here $\|\mathbf{A}\|_2$ denotes the spectral norm of $\mathbf{A}$.

Proposition 2 calls for the control of the spectral norm of $\mathbf{A}$, in addition to $\mu(\mathbf{A})$, in order to relax the sparsity constraint from (11) to (23).

In Section 4 we prove the following spectral norm bound.

Theorem 2. *Under the assumptions of Theorem 1 the matrix $\mathbf{A}$ has full rank and its spectral norm satisfies the bound*

$$\|\mathbf{A}\|_2^2 \leq \frac{2m}{np} \tag{25}$$

with probability greater than

$$1 - c_1 n(n-1)p(p-1)\sqrt{\frac{np-1}{m}}, \quad n, p \geq 2 \tag{26}$$

for some constant $c_1 > 0$.

In the case of one-shot measurements $p = 1$, we have

$$\|\mathbf{A}\|_2^2 \leq \frac{2m}{n} \tag{27}$$

with probability larger than

$$1 - \frac{c_1 n(n-1)^{3/2}}{m^{1/2}}, \quad n \geq 2.$$

Now we are ready to prove the main result for inverse Born scattering.

Theorem 3. *Let the sensors and the target be drawn randomly from the sensor and target ensembles, respectively, and consider the sensing matrix $\mathbf{A}$ of the entries defined by (8)-(9). If (13) holds, then the targets of sparsity up to*

$$s < \left(8 \ln \frac{m}{\varepsilon}\right)^{-1} \left(\chi^{\mathrm{i}} + \frac{\sqrt{2}K}{\sqrt{p}}\right)^{-2} \left(\chi^{\mathrm{s}} + \frac{\sqrt{2}K}{\sqrt{n}}\right)^{-2} \tag{28}$$

can be recovered by BP with probability greater than

$$\left(1-2\delta-c_1 n(n-1)p(p-1)\sqrt{\frac{np-1}{m}}\right)\left(1-2\varepsilon-s^{-q}\right), \quad n\geq 2, \quad p\geq 2 \tag{29}$$

for some constant $c_1>0$ *where, for* $pn\gg s$, q *can be chosen as*

$$q=\frac{\ln m-\ln\varepsilon}{72e^{1/2}\ln s}.$$

In the case of one-shot measurements $p=1$ *the probability bound (26) becomes*

$$\left(1-2\delta-\frac{c_1 n(n-1)^{3/2}}{m^{1/2}}\right)\left(1-2\varepsilon-s^{-q}\right), \quad n\geq 2. \tag{30}$$

Remark 3. *Typically the total number of grid points* m *is much larger than the target sparsity* s *and the dimension* np *of the measurement vector. In this case the probability bound (29) is close to one.*

Remark 4. *Our results can be extended to the case that the sampling angles are not independent of the incident angles by adjusting the probability (26) (and hence (29) and (30)) in the spectral norm bound. An important example is the measurement for the multistatic response data matrix with* $\tilde{\theta}_j=-\theta_j, j=1,...,n$. *Such a setting is employed in the well known and widely used multi-shot SIMO imaging scheme called MUSIC (standing for* MU*ltiple*-SI*gnal*-C*lassification) with* $p=n$ [8, 36] *(see the Conclusion).*

Previously in [17], *we have applied the compressed sensing techniques to analyze the response matrix imaging under the paraxial approximation and the assumption that the scatterers lie on a transverse plane.*

Proof. First, the coherence estimate (14) and the sparsity constraint (28) implies (23).

Now the norm bound (25) implies (24) if

$$3\left(\frac{q\ln s}{2\ln\frac{m}{\epsilon}}\right)^{1/2}+\frac{2s}{np}\leq\frac{1}{4e^{1/4}}, \quad q>1. \tag{31}$$

Hence for $np\gg s$ we can choose q in (24) to be

$$q=\frac{\ln m-\ln\varepsilon}{72\sqrt{e}\ln s}.$$

Since Theorems 1 and 2 hold with probability greater than

$$1-2\delta-c_1 n(n-1)p(p-1)\sqrt{\frac{np-1}{m}}, \quad p,n\geq 2$$

and since the target ensemble is independent of the sensor ensemble we have the bound (29) for the probability of exact recovery. The proof for the case of one-shot measurements $p=1$ is the same.

This completes the proof of Theorem 3.

□

2.2. **Exact inverse scattering.** Next we turn to the exact inverse scattering problem (6)-(7).

We work with the alternative definition of the target vector $X=(\nu_j u(\mathbf{r}_j))\in\mathbb{C}^m$ and consider only the case of one-shot measurements with $p=1$. The reason for this is that the problem then has the *appearance of linear* system

$$Y=\mathbf{A}X \tag{32}$$

where the sensing matrix $\mathbf{A}$ has the entries

$$A_{lj} = e^{-i\omega(z_j \sin\tilde{\theta}_l + x_j \cos\tilde{\theta}_l)}. \tag{33}$$

We then apply Theorem 3 with $p = 1$ and sufficiently large n to recover X with high probability. To recover ν from X, we observe that as long as $u(\mathbf{r}_{i_j}) \neq 0, \forall j$, the support of ν is the same as that of X. Indeed, the target strength $\nu = (\nu_j) \in \mathbb{C}^m$ can be recovered exactly by solving the system of nonlinear equations on the target support as follows.

Define the illumination and full field vectors at the locations of the scatterers:

$$\begin{aligned} U^{\mathrm{i}} &= (u^{\mathrm{i}}(\mathbf{r}_{i_1}), ..., u^{\mathrm{i}}(\mathbf{r}_{i_s}))^T \in \mathbb{C}^s \\ U &= (u(\mathbf{r}_{i_1}), ..., u(\mathbf{r}_{i_s}))^T \in \mathbb{C}^s. \end{aligned}$$

Let $\mathbf{G}$ be the $s \times s$ matrix

$$\mathbf{G} = [(1 - \delta_{jl}) G(\mathbf{r}_{i_j}, \mathbf{r}_{i_l})]$$

and $\mathcal{V}$ the diagonal matrix

$$\mathcal{V} = \mathrm{diag}(\nu_{i_1}, ..., \nu_{i_s}).$$

The Foldy-Lax equation (7) can be written as

$$U = U^{\mathrm{i}} + \omega^2 \mathbf{G}\mathcal{V}U \tag{34}$$

from which we obtain

$$U = \left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1} U^{\mathrm{i}} \tag{35}$$

and

$$X = \mathcal{V}U = \mathcal{V}\left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1} U^{\mathrm{i}} \tag{36}$$

provided that ω^{-2} is not an eigenvalue of $\mathbf{G}\mathcal{V}$. The exciting field then determines the scattering amplitude by (6) which yields the (nonlinear) system

$$Y = \mathbf{A}X = \mathbf{A}\mathcal{V}\left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1} U^{\mathrm{i}} \tag{37}$$

where the sensing matrix entries are given by (33).

Proposition 3. *Suppose*

$$\omega^{-2} \quad \text{is not an eigenvalue of the matrix } \mathbf{G}\mathcal{V} \tag{38}$$

and

$$U^{\mathrm{i}} \quad \text{is not orthogonal to any row vector of } \left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1}. \tag{39}$$

Then the solution $\mathcal{V}$ of (36) is given by

$$\mathcal{V} = \mathrm{diag}\left[\frac{X}{\omega^2 \mathbf{G}X + U^{\mathrm{i}}}\right] \tag{40}$$

where the division is in the entry-wise sense (Hadamard product). In this case, $\mathrm{supp}(\mathcal{V}) = \mathrm{supp}(X)$.

Proof. Note that

$$\mathcal{V}\left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1} = \left(\mathbf{I} - \omega^2 \mathcal{V}\mathbf{G}\right)^{-1} \mathcal{V}.$$

Hence eq. (36) can be written as

$$X = \left(\mathbf{I} - \omega^2 \mathcal{V}\mathbf{G}\right)^{-1} \mathcal{V}U^{\mathrm{i}}$$

or equivalently

$$\left(\mathbf{I} - \omega^2 \mathcal{V}\mathbf{G}\right) X = \mathcal{V}U^{\mathrm{i}}. \tag{41}$$

Solving (41) for the diagonal matrix $\mathcal{V}$ entry-by-entry, we obtain (40) which is well-defined if

$$\omega^2 \mathbf{G} X + U^{\mathrm{i}} \quad \text{contains no zero component.} \tag{42}$$

Since

$$\omega^2 \mathbf{G} X + U^{\mathrm{i}} = \left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1} U^{\mathrm{i}} = U$$

(42) follows from (39).

□

Corollary 1. *Condition (42) holds and hence (40) is well-defined if*

$$\omega^2 \|\mathbf{G}\mathcal{V}\| < 1/2 \tag{43}$$

where $\|\cdot\|$ *equals the maximum of the absolute row sums of the matrix corresponding to the operator norm on* L^∞*.*

Proof. Clearly $U^{\mathrm{i}} + \omega^2 \mathbf{G} X$ contains no zero entry if

$$\omega^2 \|\mathbf{G} X\| < 1 \tag{44}$$

since every component of U^{i} has modulus one. As

$$\|\mathbf{G} X\| = \|\mathbf{G}\mathcal{V}\left(\mathbf{I} - \omega^2 \mathbf{G}\mathcal{V}\right)^{-1}\|$$

(44) follows from (43). □

Theorem 4. *Suppose* $p = 1$*, (13), (38) and (39) hold. Let* X *be a BP solution for the system (32) with the matrix entries (33) according to Theorem 3. The formula (40) recovers exactly the target of sparsity*

$$s < \left(8 \ln \frac{m}{\varepsilon}\right)^{-1} \left(\chi^{\mathrm{s}} + \frac{\sqrt{2}K}{\sqrt{n}}\right)^{-2} \tag{45}$$

with probability at least as in (30) and χ^{s} *satisfies the bound (18) or (16) depending on whether* $\mathrm{supp}(f^{\mathrm{s}})$ *contains a Blind Spot or not.*

Remark 5. *The resonance frequency violating (38) is related to the transmission eigenvalue for continuous media where an analogous non-resonance condition is also needed to ensure the existence and uniqueness of the solution to the inverse scattering problem* [9, 30].

If 1 *is an eigenvalue of* $\omega^2 \mathbf{G}\mathcal{V}$*, the existence of solution for (34) requires that* U^{i} *be orthogonal to the eigenspace of* $\omega^2 \mathbf{G}\mathcal{V}$ *corresponding to* 1*. Then other physical constraints (such as the minimum energy solution) need to be taken into account in order to obtain a unique solution.*

The simplest example for resonance is this: Two point scatterers have the strengths ν_{i_1}, ν_{i_2} *such that* $\mathrm{sign}(\nu_{i_1}) = \mathrm{sign}(\nu_{i_2}) = \mathrm{sign}(G^*(\mathbf{r}_{i_1}, \mathbf{r}_{i_2}))$*. Then*

$$\mathbf{G}\mathcal{V} = \begin{bmatrix} 0 & |\nu_{i_1} G(\mathbf{r}_{i_1}, \mathbf{r}_{i_2})| \\ |\nu_{i_2} G(\mathbf{r}_{i_1}, \mathbf{r}_{i_2})| & 0 \end{bmatrix}$$

is real and symmetric and has the positive eigenvalue $\sqrt{|\nu_{i_1}\nu_{i_2}|}|G(\mathbf{r}_{i_1}, \mathbf{r}_{i_2})|$*.*

Remark 6. *In view of (35), (39) means that* U *has no zero component. In other words, the target is not shadowed by itself in any way.*

Since the negation of (39) is an algebraic constraint, (39) are satisfied almost surely in the target ensemble under (38).

2.3. **Noisy measurement.** Finally we consider the case where the measurement data contain errors or noise. Using a relaxation scheme in compressed sensing [13, 38] we derive an error bound and a sufficient condition under which the support of the reconstruction is exactly the same as the original, both in the case of nonlinear inverse scattering.

Let $X = (\nu_j u(\mathbf{r}_{i_j})) \in \mathbb{C}^m$ and Y as in (37). Consider for definiteness the standard model of additive noise

$$Y = \mathbf{A}X + E$$

where E is the noise with $\|E\|_2 \leq \varepsilon$.

Since $A(\hat{\mathbf{r}}, \mathbf{d})$ is an analytic function of both $\hat{\mathbf{r}}$ and $\mathbf{d}$ and hence for the given noisy data, in general no solution exists to the inverse scattering problem. Even if a solution does exist, it does not depend continuously on the measured data in any reasonable norm.

To deal with the problem of ill-posedness we consider, instead of the Tikhonov regularization, the L^1-regularization

$$\min_Z \frac{1}{2}\|Y - \mathbf{A}Z\|_2^2 + 2\varepsilon\|Z\|_1. \tag{46}$$

Let $\hat{X}$ be the minimizer of (46). Define the reconstruction of $\mathcal{V}$ to be

$$\hat{\mathcal{V}} = \text{diag}\left[\frac{\hat{X}}{U^{\text{i}} + \omega^2\mathbf{G}\hat{X}}\right] \tag{47}$$

in analogy to (40).

We prove the following result in Section 5.

Theorem 5. *Assume*

$$\mu s \leq 1/3 \tag{48}$$

and

$$\omega^2\|\mathbf{G}\mathcal{V}\| < \frac{1-(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\|}{2-(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\|} \quad \left(< \frac{1}{2}\right). \tag{49}$$

Then (47) is well-defined and satisfies the error bound:

$$\|\mathcal{V} - \hat{\mathcal{V}}\| \leq \frac{2\left(1+\omega^2\|\mathbf{G}\|\|\mathcal{V}\|\right)(3+\sqrt{3/2})\varepsilon}{b_0(b_0 - \omega^2(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\|)}, \quad b_0 \equiv \frac{1-2\omega^2\|\mathbf{G}\mathcal{V}\|}{1-\omega^2\|\mathbf{G}\mathcal{V}\|}. \tag{50}$$

Moreover, $\text{supp}(\hat{\mathcal{V}}) = \text{supp}(\hat{X}) \subset \text{supp}(X)$. *On the other hand, if*

$$\omega^2\|\mathbf{G}\mathcal{V}\| \quad < \quad \frac{1-(3+\sqrt{3/2})\varepsilon\|\mathcal{V}^{-1}\|}{2-(3+\sqrt{3/2})\varepsilon\|\mathcal{V}^{-1}\|} \tag{51}$$

then $\text{supp}(\hat{X}) = \text{supp}(X)$. *Therefore under (49) and (51),* $\text{supp}(\hat{\mathcal{V}}) = \text{supp}(\mathcal{V})$, *i.e. the support of the target is perfectly recovered.*

Remark 7. *Condition (48) is slightly stronger than (11) and hence the OMP algorithm can be used to solve (46)* [38].

Condition (49), (51) and (43) all say in various way that the scatterers are either weak or far apart.

3. PROOF OF THEOREM 1: COHERENCE BOUND

Proof. Denote $\hat{\mathbf{r}}_j = (\cos\tilde{\theta}_j, \sin\tilde{\theta}_j), \mathbf{d}_k = (\cos\theta_k, \sin\theta_k)$.

The pairwise coherence has the form

$$\frac{1}{pn}\left|\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')}\sum_{j=1}^{n} e^{i\omega\hat{\mathbf{r}}_j\cdot(\mathbf{r}-\mathbf{r}')}\right| \tag{52}$$

where $\mathbf{r}, \mathbf{r}'$ are two distinct points in the lattice $\mathcal{M}$. Note that the two summations in (52) are of the same type.

Consider the first summation. Let

$$X_k = \cos\left(\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')\right), \quad Y_k = \sin\left(\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')\right)$$

and

$$S_p = \sum_{k=1}^{p} X_k, \quad T_p = \sum_{k=1}^{p} Y_k.$$

Then the first summation in (52) can be bounded by

$$\left|\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')}\right| \leq |S_p - \mathbb{E}S_p| + |T_p - \mathbb{E}T_p| + |\mathbb{E}(S_p + iT_p)|$$

We recall the Hoeffding inequality [20].

Proposition 4. *Let $X_1, ..., X_p$ be independent random variables. Assume that $X_l \in [a_l, b_l], l = 1, ..., p$ almost surely. Then we have*

$$\mathbb{P}\left[|S_p - \mathbb{E}S_p| \geq pt\right] \leq 2\exp\left[-\frac{2p^2t^2}{\sum_{l=1}^{p}(b_l - a_l)^2}\right] \tag{53}$$

for all positive values of t.

We apply the Hoeffding inequality to both S_p and T_p. To this end, we have $b_l - a_l = 2, \forall l = 1, ..., p$. Set

$$t = K/\sqrt{p}, \quad K > 0.$$

Then we obtain

$$\mathbb{P}\left[p^{-1}|S_p - \mathbb{E}S_p| \geq K/\sqrt{p}\right] \leq 2e^{-K^2/2} \tag{54}$$

$$\mathbb{P}\left[p^{-1}|T_p - \mathbb{E}T_p| \geq K/\sqrt{p}\right] \leq 2e^{-K^2/2}. \tag{55}$$

Note that the quantities S_p, T_p depend on $\mathbf{r} - \mathbf{r}' = (x_i - x_j, z_i - z_j)$. We use (54)-(55) and the union bound to obtain

$$\mathbb{P}\left[\max_{i\neq j} p^{-1}|S_p - \mathbb{E}S_p| \geq K/\sqrt{p}\right] \leq 2(m-1)e^{-K^2/2}$$

$$\mathbb{P}\left[\max_{i\neq j} p^{-1}|T_p - \mathbb{E}T_p| \geq K/\sqrt{p}\right] \leq 2(m-1)e^{-K^2/2}$$

Hence,

$$\mathbb{P}\left[\max_{i\neq j} p^{-1}\left|\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')} - \mathbb{E}\left[\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')}\right]\right| < \sqrt{2}K/\sqrt{p}\right] > (1 - 2(m-1)e^{-K^2/2})^2. \tag{56}$$

Similarly we have for the second summation in (52)

$$\mathbb{P}\left[\max_{i\neq j} n^{-1}\left|\sum_{j=1}^{n} e^{i\omega\hat{\mathbf{r}}_j\cdot(\mathbf{r}-\mathbf{r}')} - \mathbb{E}\left[\sum_{j=1}^{n} e^{i\omega\hat{\mathbf{r}}_j\cdot(\mathbf{r}-\mathbf{r}')}\right]\right| < \sqrt{2}K/\sqrt{n}\right] > \left(1-2(m-1)e^{-K^2/2}\right)^2. \tag{57}$$

By (13) the right hand side of (57)-(56) is greater than $(1-\delta)^2$.

Let us calculate $\mathbb{E}\left[\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')}\right]$. If f^{s} is the uniform distribution over $[-\pi,\pi]$ or $[-\pi/2,\pi/2]$ then

$$\mathbb{E}\left[\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')}\right] = pJ_0(\omega|\mathbf{r}-\mathbf{r}'|)$$

where J_0 is the zeroth order Bessel function. In general, the exact expression is not available but we are concerned only with the asymptotic for $\omega|\mathbf{r}-\mathbf{r}'|\gg 1$.

The Bessel function has the asymptotic

$$J_0(\omega r) = \sqrt{\frac{2}{\pi\omega r}}\left\{\cos(\omega r-\pi/4)+\mathcal{O}((\omega r)^{-1})\right\}, \quad \omega r\gg 1. \tag{58}$$

That is, for $\omega\ell\gg 1$ there exists a constant $c>0$ such that

$$J_0(\omega|\mathbf{r}-\mathbf{r}'|) < \frac{c}{\sqrt{\omega\ell}}, \quad \forall \mathbf{r},\mathbf{r}'\in\mathcal{M}, \quad \mathbf{r}\neq\mathbf{r}' \tag{59}$$

In general,

$$\frac{1}{p}\mathbb{E}\left[\sum_{k=1}^{p} e^{i\omega\mathbf{d}_k\cdot(\mathbf{r}-\mathbf{r}')}\right] = \int_0^{2\pi} e^{i\omega\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}')} f^{\mathrm{i}}(\theta)d\theta \tag{60}$$

which is the Herglotz wave function with kernel f^{i} in two dimensions. By assumption on f^{i} (60) is a finite sum of integrals of the form

$$\int_a^b e^{i\omega\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}')} f^{\mathrm{i}}(\theta)d\theta \tag{61}$$

with $f^{\mathrm{i}}\neq 0$ in (a,b) whose asymptotic for $\omega\ell\gg 1$ can be analyzed by the method of stationary phase (Theorem XI. 14 and XI. 15 of [34]).

Proposition 5. *Let* $g_{\mathbf{r},\mathbf{r}'}(\theta)=\mathbf{d}\cdot(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ *which is in* $C^\infty([-\pi,\pi]), \forall\mathbf{r},\mathbf{r}'\in\mathcal{M}$.

(i) Suppose $\frac{d}{d\theta}g_{\mathbf{r},\mathbf{r}'}(\theta)\neq 0, \forall\theta\in[a,b], \forall\mathbf{r},\mathbf{r}'\in\mathcal{M}$. *Then for all* $f^{\mathrm{i}}\in C_0^h([a,b])$

$$\left|\int e^{i\omega|\mathbf{r}-\mathbf{r}'|g_{\mathbf{r},\mathbf{r}'}(\theta)} f^{\mathrm{i}}(\theta)d\theta\right| \leq c_h(1+\omega|\mathbf{r}-\mathbf{r}'|)^{-h}\|f^{\mathrm{i}}\|_{h,\infty} \tag{62}$$

for some constant c_h *independent of* f^{i}. *Moreover, since* $\{g_{\mathbf{r},\mathbf{r}'}:\mathbf{r},\mathbf{r}'\in\mathcal{M}\}$ *is a compact subset of* $C^{h+1}([a,b])$, *the constant* c *can be chosen uniformly for all* $\mathbf{r},\mathbf{r}'\in\mathcal{M}$.

(ii) Suppose $\frac{d}{d\theta}g_{\mathbf{r},\mathbf{r}'}(\theta)$ *vanishes at* $\theta_*\in(a,b)$. *Since* $\frac{d^2}{d\theta^2}g_{\mathbf{r},\mathbf{r}'}(\theta_*)\neq 0$, *there exists a constant* $c_t, t>1/2$ *such that*

$$\left|\int e^{i\omega|\mathbf{r}-\mathbf{r}'|g_{\mathbf{r},\mathbf{r}'}(\theta)} f^{\mathrm{i}}(\theta)d\theta\right| \leq c_t(1+\omega|\mathbf{r}-\mathbf{r}'|)^{-1/2}\|f^{\mathrm{i}}\|_{t,\infty} \tag{63}$$

where the constant c_t *is independent of* $\mathbf{r},\mathbf{r}'\in\mathcal{M}$.

Note that the condition

$$\frac{d}{d\theta} g_{\mathbf{r},\mathbf{r}'}(\theta) \neq 0, \quad \theta \in [a,b], \quad \forall \mathbf{r},\mathbf{r}' \in \mathcal{M}$$

is the same as saying that (a,b) does not contain any Blind Spot. Applying Proposition 5 to

$$\chi^{\mathrm{i}} \;= \left| \int \left[e^{i\omega \mathbf{d}\cdot(\mathbf{r}-\mathbf{r}')} \right] f^{\mathrm{i}}(\theta) d\theta \right|, \quad \chi^{\mathrm{s}} \;= \left| \int \left[e^{i\omega \tilde{\mathbf{d}}\cdot(\mathbf{r}-\mathbf{r}')} \right] f^{\mathrm{s}} d\theta \right|,$$

using (56)-(57) and the identity

$$PQ = (P-\bar{P})(Q-\bar{Q}) + \bar{P}(Q-\bar{Q}) + \bar{Q}(P-\bar{P}) + \bar{P}\bar{Q}$$

we obtain (14) with probability greater than $(1-\delta)^2$.

□

3.1. **Three dimension.** In three dimensions, the scattering amplitude has the form

$$A(\hat{\mathbf{r}},\mathbf{d}) \;=\; \frac{\omega^2}{4\pi} \sum_{j=1}^{m} \nu_j u(\mathbf{r}_j) e^{-i\omega \mathbf{r}_j \cdot \hat{\mathbf{r}}} \tag{64}$$

where the sampling direction $\hat{\mathbf{r}} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ can be parametrized by the polar angles $\tilde{\theta}, \tilde{\phi}$ as

$$\tilde{\alpha} = \cos\tilde{\theta}\cos\tilde{\phi}, \quad \tilde{\beta} = \cos\tilde{\theta}\sin\tilde{\phi}, \quad \tilde{\gamma} = \sin\tilde{\theta} \tag{65}$$

Let $\mathbf{d} = (\alpha, \beta, \gamma)$ be parameterized by the angles θ, ϕ as

$$\alpha = \cos\theta\cos\phi, \quad \beta = \cos\theta\sin\phi, \quad \gamma = \sin\theta. \tag{66}$$

The pairwise coherence has the form

$$\frac{1}{pn} \sum_{k=1}^{p} e^{i\omega(\alpha_k,\beta_k,\gamma_k)\cdot(\mathbf{r}-\mathbf{r}')} \sum_{j=1}^{n} e^{i\omega(\tilde{\alpha}_j,\tilde{\beta}_j,\tilde{\gamma}_j)\cdot(\mathbf{r}-\mathbf{r}')} \tag{67}$$

where $(\alpha_k, \beta_k, \gamma_k), k = 1, ..., p$ and $(\tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}_j), j = 1, ..., n$ are independently and identically distributed in the *unit* sphere according to $f^{\mathrm{i}}(\theta,\phi)$ and $f^{\mathrm{s}}(\theta,\phi)$, respectively.

The main difference between two and three dimensions is in evaluating the expectation of $p^{-1}\sum_{k=1}^{p} e^{i\omega(\alpha_k,\beta_k,\gamma_k)\cdot(\mathbf{r}-\mathbf{r}')}$ and $n^{-1}\sum_{j=1}^{n} e^{i\omega(\tilde{\alpha}_j,\tilde{\beta}_j,\tilde{\gamma}_j)\cdot(\mathbf{r}-\mathbf{r}')}$ which amounts to calculating the integrals

$$\int_{-\pi/2}^{\pi/2} d\theta f_1^{\mathrm{i}}(\theta+\theta_0)\cos\theta \exp\left[i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta\right] \tag{68}$$

$$\int_{-\pi/2}^{\pi/2} d\theta f_1^{\mathrm{s}}(\theta+\theta_0)\cos\theta \exp\left[i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta\right] \tag{69}$$

for some θ_0 depending on $\mathbf{r}-\mathbf{r}'$ where f_1^{i} and f_1^{s} are the marginal density functions

$$\begin{aligned} f_1^{\mathrm{i}}(\theta) &= \int_{-\pi}^{\pi} d\phi f^{\mathrm{i}}(\theta,\phi) \\ f_1^{\mathrm{s}}(\theta) &= \int_{-\pi}^{\pi} d\phi f^{\mathrm{s}}(\theta,\phi) \end{aligned}$$

If $f_1^{\mathrm{i}} = f_1^{\mathrm{s}} = 1/\pi$, the integrals (68) and (69) become

$$\frac{2\sin(\omega|\mathbf{r}-\mathbf{r}'|)}{\omega|\mathbf{r}-\mathbf{r}'|} = \mathcal{O}(\frac{1}{\omega\ell}), \quad \omega\ell \gg 1.$$

For the general case, integrating by parts with (68) and (69) produces

$$\frac{i}{\omega|\mathbf{r}-\mathbf{r}'|}\left[f_1^{\mathrm{i}}(\theta+\theta_0)e^{i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta}\Big|_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}e^{i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta}\frac{d}{d\theta}f_1^{\mathrm{i}}(\theta+\theta_0)d\theta\right] \tag{70}$$

$$\frac{i}{\omega|\mathbf{r}-\mathbf{r}'|}\left[f_1^{\mathrm{s}}(\theta+\theta_0)e^{i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta}\Big|_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}e^{i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta}\frac{d}{d\theta}f_1^{\mathrm{s}}(\theta+\theta_0)d\theta\right] \tag{71}$$

from which we obtain the bound

$$\left|\int_{-\pi/2}^{\pi/2}d\theta f_1^{\mathrm{i}}(\theta+\theta_0)\cos\theta\exp\left[i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta\right]\right|\leq\frac{c}{1+\omega\ell}\|f^{\mathrm{i}}\|_{1,\infty} \tag{72}$$

$$\left|\int_{-\pi/2}^{\pi/2}d\theta f_1^{\mathrm{s}}(\theta+\theta_0)\cos\theta\exp\left[i\omega|\mathbf{r}-\mathbf{r}'|\sin\theta\right]\right|\leq\frac{c}{1+\omega\ell}\|f^{\mathrm{s}}\|_{1,\infty}. \tag{73}$$

4. PROOF OF THEOREM 2: SPECTRAL NORM BOUND

Proof. For the proof, it suffices to show that the matrix $\mathbf{A}$ satisfies

$$\|\frac{np}{m}\mathbf{A}\mathbf{A}^*-\mathbf{I}_{np}\|_2<1 \tag{74}$$

where $\mathbf{I}_{np}$ is the $np\times np$ identity matrix with the corresponding probability bound (26). By the Gershgorin circle theorem, (74) would in turn follow from

$$\mu\left(\sqrt{\frac{np}{m}}\mathbf{A}^*\right)<\frac{1}{np-1} \tag{75}$$

since the diagonal elements of $\frac{np}{m}\mathbf{A}\mathbf{A}^*$ are unity.

The pairwise coherence amounts to calculating the expression

$$\frac{np}{m}\left|\sum_{l=1}^{m}A_{jl}A_{li}^*\right| = \frac{1}{m}\left|\sum_{l=1}^{m}e^{-i\omega(z_l\sin\theta+x_l\cos\theta)}e^{i\omega(z_l\sin\theta'+x_l\cos\theta')}e^{i\omega(z_l\sin\theta+x_l\cos\theta)}e^{-i\omega(z_l\sin\theta'+x_l\cos\theta')}\right|$$

Summing over $\mathbf{r}_l, l=1,...,m$ results in finite geometric series in the longitudinal and transverse coordinates since they are equally spaced. We obtain

$$\begin{aligned}\frac{np}{m}\left|\sum_{l=1}^{m}A_{jl}A_{li}^*\right| &= \frac{1}{m}\left|\frac{e^{i\omega\ell(\cos\theta'-\cos\theta+\cos\tilde{\theta}-\cos\tilde{\theta}')\sqrt{m}/2}-e^{-i\omega(\cos\theta'-\cos\theta+\cos\tilde{\theta}-\cos\tilde{\theta}')\sqrt{m}/2}}{e^{i\omega\ell(\cos\theta'-\cos\theta+\cos\tilde{\theta}-\cos\tilde{\theta}')}-1}\right| \\ &\quad\times\left|\frac{e^{i\omega\ell(\sin\theta'-\sin\theta+\sin\tilde{\theta}-\sin\tilde{\theta})\sqrt{m}/2}-e^{-i\omega(\sin\theta'-\sin\theta+\sin\tilde{\theta}-\sin\tilde{\theta}')\sqrt{m}/2}}{e^{i\omega\ell(\sin\theta'-\sin\theta+\sin\tilde{\theta}-\sin\tilde{\theta}')}-1}\right|.\end{aligned} \tag{76}$$

Using the identity $|1-e^{i\phi}|=2|\sin{(\phi/2)}|$ we then obtain

$$\begin{aligned}\frac{np}{m}\left|\sum_{l=1}^{m}A_{jl}A_{li}^*\right| &= \frac{1}{m}\frac{\left|\sin\left[\omega\ell(\cos\theta'-\cos\theta+\cos\tilde{\theta}-\cos\tilde{\theta}')\sqrt{m}/2\right]\right|}{\left|\sin\left[\omega\ell(\cos\theta'-\cos\theta+\cos\tilde{\theta}-\cos\tilde{\theta}')/2\right]\right|} \\ &\quad\times\frac{\left|\sin\left[\omega\ell(\sin\theta'-\sin\theta+\sin\tilde{\theta}-\sin\tilde{\theta}')\sqrt{m}/2\right]\right|}{\left|\sin\left[\omega\ell(\sin\theta'-\sin\theta+\sin\tilde{\theta}-\sin\tilde{\theta}')/2\right]\right|}\end{aligned} \tag{77}$$

Since θ, θ' are independently and identically distributed according to f^{i}, the sine and cosine of these variables have the density functions

$$g(t) = \frac{1}{\sqrt{1-t^2}} f^{\mathrm{i}}(\arcsin t) \quad \text{and} \quad \frac{1}{\sqrt{1-t^2}} f^{\mathrm{i}}(\arccos t),$$

respectively. Similarly the sine and cosine of $\tilde{\theta}, \tilde{\theta}'$ have the density functions

$$g(t) = \frac{1}{\sqrt{1-t^2}} f^{\mathrm{s}}(\arcsin t) \quad \text{and} \quad \frac{1}{\sqrt{1-t^2}} f^{\mathrm{s}}(\arccos t),$$

respectively.

Hence the random variables

$$\begin{aligned} Z_1 \quad &= \omega\ell(\cos\theta' - \cos\theta + \cos\tilde{\theta} - \cos\tilde{\theta}')/2 \quad \in [-2\omega\ell, 2\omega\ell] \\ Z_2 \quad &= \omega\ell(\sin\theta' - \sin\theta + \sin\tilde{\theta} - \sin\tilde{\theta}')/2 \quad \in [-2\omega\ell, 2\omega\ell] \end{aligned}$$

have the density function

$$f_{Z_1}, f_{Z_2} = \frac{1}{\omega\ell}(g * g * g * g)(\frac{2z}{\omega\ell}).$$

Since $g * g * g * g$ is bounded in $[-4, 4]$, we have

$$\|f_{Z_1}\|_\infty, \quad \|f_{Z_2}\|_\infty \le \frac{c_0}{\omega\ell}, \quad \omega\ell \gg 1$$

for some constant $c_0 > 0$.

Define

$$\zeta = \min_{\theta,\theta',\tilde{\theta},\tilde{\theta}'} \min_{k\in\mathbb{Z}} \{|Z_1 - \pi k|, |Z_2 - \pi k|\} \tag{78}$$

and note

$$\sin\zeta > \frac{2\zeta}{\pi}, \quad \zeta \in (0, \pi/2).$$

Hence the probability that $\{\zeta > b\}$ for small $b > 0$ is larger than

$$(1 - c_1 b)^{n(n-1)p(p-1)} > 1 - c_1 n(n-1)p(p-1)b$$

where the power $n(n-1)p(p-1)$ accounts for the number of different pairs of random variables involved in (78).

By the choice

$$b = \sqrt{\frac{np-1}{m}}$$

we deduce that

$$\mu\left(\sqrt{\frac{n}{m}}\mathbf{A}^*\right) < \frac{1}{mb^2} = \frac{1}{np-1}$$

with probability larger than

$$1 - c_1 n(n-1)p(p-1)b = 1 - c_1 n(n-1)p(p-1)\sqrt{\frac{np-1}{m}}.$$

In the one-shot case $p = 1$, (78) becomes

$$\zeta = \min_{\tilde{\theta},\tilde{\theta}'} \min_{k\in\mathbb{Z}} \{|Z_1 - \pi k|, |Z_2 - \pi k|\}. \tag{79}$$

Hence the probability that $\{\zeta > b\}$ for small $b > 0$ is larger than

$$(1 - c_1 b)^{n(n-1)} > 1 - c_1 n(n-1)b.$$

With

$$b = \sqrt{\frac{n-1}{m}}$$

it follows that

$$\mu\left(\sqrt{\frac{n}{m}}\mathbf{A}^*\right) < \frac{1}{mb^2} = \frac{1}{np-1}$$

with probability larger than

$$1 - \frac{c_1 n(n-1)^{3/2}}{m^{1/2}}.$$

□

5. Proof of Theorem 5: Stability

The starting point of the proof is the following result, due to Tropp [38], concerning the error bound and the recoverability of the target support in the presence of noise.

Proposition 6. [38] *Suppose $\mu s \le 1/3$. Then the minimizer $\hat{X}$ of (46) is unique and its support is contained in* supp(X). *Moreover,*

$$\|\hat{X} - X\|_\infty \le \left(3 + \sqrt{3/2}\right)\varepsilon. \tag{80}$$

Next, we give an estimate for the smallest component of the exciting field vector $U = (1 - \omega^2\mathbf{G}\mathcal{V})^{-1}U^{\mathrm{i}}$.

Proposition 7. *If*

$$\omega^2\|\mathbf{G}\mathcal{V}\| < 1/2 \tag{81}$$

then

$$\left\|\frac{1}{\left(\mathbf{I} - \omega^2\mathbf{G}\mathcal{V}\right)^{-1}U^{\mathrm{i}}}\right\|_\infty \le \frac{1-\omega^2\|\mathbf{G}\mathcal{V}\|}{1-2\omega^2\|\mathbf{G}\mathcal{V}\|} \equiv \frac{1}{b_0}. \tag{82}$$

Here for any vector V, V^{-1} denotes the vector whose entries are the reciprocal of those of V.

Proof. We write

$$\left(\mathbf{I} - \omega^2\mathbf{G}\mathcal{V}\right)^{-1}U^{\mathrm{i}} = U^{\mathrm{i}} \odot (1 \oplus R)$$

with

$$R = (U^{\mathrm{i}})^{-1} \odot \left(\omega^2\mathbf{G}\mathcal{V}U^{\mathrm{i}} + (\omega^2\mathbf{G}\mathcal{V})^2U^{\mathrm{i}} + \ldots\right)$$

which converges under (81). Here $\odot$ and $\oplus$ denote the entrywise (Hadamard) product and sum, respectively, of two vectors. Hence

$$\left\|\frac{1}{\left(\mathbf{I} - \omega^2\mathbf{G}\mathcal{V}\right)^{-1}U^{\mathrm{i}}}\right\|_\infty \le \left\|\frac{1}{U^{\mathrm{i}} \odot (1 \oplus R)}\right\|_\infty \le \frac{1}{1-\|R\|_\infty}. \tag{83}$$

We also have

$$\|R\|_\infty \le \left(\omega^2\|\mathbf{G}\mathcal{V}\| + \omega^4\|\mathbf{G}\mathcal{V}\|^2 + \ldots\right) = \frac{\omega^2\|\mathbf{G}\mathcal{V}\|}{1-\omega^2\|\mathbf{G}\mathcal{V}\|}. \tag{84}$$

Substituting (84) into (83) we obtain the claimed bound (82). □

From the Foldy-Lax equation (35) and (82) we have the following lower bound on the exciting field vector U

$$\left\|U^{-1}\right\|_\infty \le 1/b_0 \tag{85}$$

and hence $b_0 \le |u(\mathbf{r}_{i_j})|, \forall j$.

Corollary 2. *Suppose* $\mu s \le 1/3$ *and*

$$b_0 > \left(3+\sqrt{3/2}\right)\varepsilon\|\mathcal{V}^{-1}\|. \tag{86}$$

Then $\operatorname{supp}(\hat{X}) = \operatorname{supp}(X)$.

Proof. This follows immediately from the fact

$$\min_j |X_j| = \min_j |\nu_{i_j} u(\mathbf{r}_{i_j})| \ge \frac{b_0}{\|\mathcal{V}^{-1}\|} > (3+\sqrt{3/2})\varepsilon$$

and Proposition 6. □

Proposition 8. *The vector* $U^{\mathrm{i}} + \omega^2\mathbf{G}\hat{X}$ *contains no zero entry if* $\omega^2\|\mathbf{G}\hat{X}\|_\infty < 1$. *In particular, this is true for the minimizer* $\hat{X}$ *of Proposition 6 under the additional assumption*

$$b_0 > \omega^2(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\|. \tag{87}$$

In this case, $\operatorname{supp}(\hat{\mathcal{V}}) = \operatorname{supp}(\hat{X})$.

Proof. The following calculation is straightforward

$$\|\mathbf{G}\hat{X} - \mathbf{G}X\|_\infty \quad \le \|\mathbf{G}\|\|\hat{X} - X\|_\infty \quad \le (3+\sqrt{3/2})\varepsilon\|\mathbf{G}\|. \tag{88}$$

Moreover, since

$$\mathbf{G}X = \mathbf{G}\mathcal{V}(\mathbf{I} - \omega^2\mathbf{G}\mathcal{V})^{-1}U^{\mathrm{i}}$$

we have the estimate

$$\|\mathbf{G}X\|_\infty \le \frac{\|\mathbf{G}\mathcal{V}\|}{1-\omega^2\|\mathbf{G}\mathcal{V}\|}. \tag{89}$$

Hence

$$\begin{aligned}
\omega^2\|\mathbf{G}\hat{X}\|_\infty &\le \omega^2\|\mathbf{G}X\|_\infty + \omega^2\|\mathbf{G}\hat{X} - \mathbf{G}X\|_\infty \\
&\le \frac{\omega^2\|\mathbf{G}\mathcal{V}\|}{1-\omega^2\|\mathbf{G}\mathcal{V}\|} + \omega^2(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\| \\
&= 1 - b_0 + \omega^2(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\| < 1 \qquad (90)
\end{aligned}$$

under the additional condition (87). □

The proof of Theorem 5 can now be completed as follows.

Proof. First of all, (49) is equivalent to (87) and by Proposition 8 the formula (47) is well-defined.

Subtracting (40) from (47) we can estimate as follows:

$$\begin{aligned}
\|\mathcal{V} - \hat{\mathcal{V}}\| &= \left\| \frac{U^{\mathrm{i}} \odot (X-\hat{X}) + \omega^2(X-\hat{X}) \odot \mathbf{G}X - \omega^2 X \odot \mathbf{G}(X-\hat{X})}{(U^{\mathrm{i}} + \omega^2\mathbf{G}\hat{X})(U^{\mathrm{i}} + \omega^2\mathbf{G}X)} \right\| \\
&\le \frac{\left(1+\omega^2\|\mathbf{G}X\|_\infty\right)\|X-\hat{X}\|_\infty + \omega^2\|X\|_\infty\|\mathbf{G}X - \mathbf{G}\hat{X}\|_\infty}{1-\omega^2\|\mathbf{G}\hat{X}\|_\infty} \times \|U^{-1}\|_\infty
\end{aligned}$$

where we have used the identity

$$U = \omega^2\mathbf{G}X + U^{\mathrm{i}}. \tag{91}$$

By (81) and (89) we find that

$$\omega^2\|\mathbf{G}X\|_\infty \quad < \quad 1 \tag{92}$$

And, since $X = (\nu_j u(\mathbf{r}_j))$,

$$\|X\|_\infty \le \|\mathcal{V}\|\|U\|_\infty.$$

This, (85), (88) and Proposition 6 lead to the bound

$$\|\mathcal{V}-\hat{\mathcal{V}}\| \le \frac{(2+\omega^2\|\mathbf{G}\|\|\mathcal{V}\|\|U\|_\infty)(3+\sqrt{3/2})\varepsilon}{b_0(b_0-\omega^2(3+\sqrt{3/2})\varepsilon\|\mathbf{G}\|)}. \tag{93}$$

In view of (91) and (89) we have the following bound

$$\|U\|_\infty \quad \le 1+\omega^2\|\mathbf{G}X\|_\infty \quad \le \frac{1}{1-\omega^2\|\mathbf{G}\mathcal{V}\|} < 2. \tag{94}$$

The claimed result (50) now follows from (93) and (94).

Since (51) is equivalent with (86) it follows from Corollary 2, Propositions 8 and 3 that $\text{supp}(\hat{\mathcal{V}}) = \text{supp}(\mathcal{V})$. □

6. Conclusion

We have analyzed the SIMO inverse scattering problem by the compressed sensing techniques to hopefully shed new light on this problem with distinguished history [9, 24, 28, 34]. We have obtained three main results: Theorem 3 concerns the recoverability by BP with multiple-shot SIMO measurements under the Born approximation, Theorem 4 addresses the recoverability by BP with single-shot SIMO measurements and Theorem 5 asserts stability to noisy for weak or widely separated scatterers under a stronger sparsity constraint. .

Three issues in the analysis merit further study: the notion of Blind Spots that may slow down the approach to the high frequency asymptotic (Theorem 3 and 4), the notion of resonance frequency and the notion of self-induced shadow which can prevent the exact recovery of the target support (Theorem 4).

A main limitation to our approach is the assumption of point scatterers that are a finite (albeit high) dimensional target. Also, the reconstruction succeeds only probabilistically. These limitations are intrinsic to the current formulation of the compressed sensing theory which is still evolving and in this regard the present paper is only a first step in the new direction.

On the other hand, the compressed sensing approach applies to any dimensions, is constructive and treats the uniqueness and the reconstruction in a unified way. Indeed, the main advantage of this approach is an explicit and efficient method (i.e. the Basis Pursuit with the sensing matrix determined by the SIMO-sensor ensemble) for reconstructing the scatterers from the scattering amplitude. Moverover, the aperture can be rather arbitrary and the dimension of measurement can be as low as comparable to the target sparsity (up to a $\log(m)$-factor).

It may be worthwhile to compare our imaging method with the MUSIC algorithm which employs multiple sensors to collect the $n\times n$ multistatic response data matrix where n is the number of transmitters/receivers [8, 36]. When the measurement is carried out in the far field, the (l,j)-entry of the response matrix is the measured scattering amplitude for the sampling direction l and the incident direction j. It is not known if MUSIC can recover the target support exactly for nonlinear inverse scattering. Only the case for the Born approximation has been shown capable of exact recovery of the target *support* in the absence of noise [21] (see the corrected argument in Theorem 4.1, [22]). And the estimate for the required dimension of the measurement for the exact recovery is hardly optimal. This result should be compared to Theorem 3 with $p=n$ and $\theta_j=-\tilde{\theta}_j, j=1,...,n$, in particular the sparsity constraint (28) for compressed sensing versus the necessary condition $n>s$ for MUSIC.

Finally we study the single-input-single-output (SISO) compressive imaging technique which requires a different approach in a separate paper [15] which also contains a numerical, comparative study of the performances of the SISO and SIMO schemes.

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