Math 133: Homework 5

Prepared by Gregory Shinault
gshinault@math.ucdavis.edu

2.9 Consider a two-period stochastic volatility, random interest rate model.

(a) Determine the risk-neutral probabilities

$$\tilde{\mathbb{P}}(HH), \tilde{\mathbb{P}}(HT), \tilde{\mathbb{P}}(TH), \tilde{\mathbb{P}}(TT)$$

such that the time-zero value of an option that pays off V_2 at time two is giaven by the risk-neutral pricing formula.

Solution. We find these probabilities by breaking the model into groupings of one-period binomial models and applying what we know from Chapter 1. For example, suppose the first coin toss is H. Then we have

$$\tilde{\mathbb{P}}(HH|\omega_1 = H) = \frac{1 + r_1(H) - d_1(H)}{u_1(H) - d_1(H)} = \frac{1 + 1/4 - 1}{3/2 - 1} = \frac{1}{2}.$$

Using conditional probability, $\tilde{\mathbb{P}}(HH) = \tilde{\mathbb{P}}(HH|\omega_1 = H)\tilde{\mathbb{P}}(\omega_1 = H)$. Since $\tilde{\mathbb{P}}(\omega_1 = H)$ is just the $\tilde{p}$ from our familiar binomial model, we can conclude $\tilde{\mathbb{P}}(HH) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Proceeding similarly for the other cases, we find

$$\begin{aligned}
\tilde{\mathbb{P}}(HT) &= \tilde{p} \cdot \frac{u_1(H) - 1 - r_1(H)}{u_1(H) - d_1(H)} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
\tilde{\mathbb{P}}(TH) &= \tilde{q} \cdot \frac{1 + r_1(T) - d_1(T)}{u_1(T) - d_1(T)} = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12} \\
\tilde{\mathbb{P}}(TT) &= \tilde{q} \cdot \frac{u_1(T) - 1 - r_1(T)}{u_1(T) - d_1(T)} = \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}
\end{aligned}$$

The computations are complete. □

(b) Let $V_2 = (S_2 - 7)^+$. Compute V_0, $V_1(H)$, and $V_1(T)$.

Solution. We begin by computing $V_1(H)$. We view the model as a one-period and use familiar results.

$$\begin{aligned}
V_1(H) &= \frac{1}{1 + r_1(H)} \left(\tilde{\mathbb{P}}(HH|\omega_1 = H)V_2(HH) + \tilde{\mathbb{P}}(HT|\omega_1 = H)V_2(HT) \right) \\
&= \frac{1}{1 + 1/4} \left(\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 1 \right) = \frac{12}{5}
\end{aligned}$$

Similar computations will show that $V_1(T) = 5/9$. We can now compute V_0:

$$V_0 = \frac{1}{1 + 1/4} \left(\tilde{p} V_1(H) + \tilde{q} V_1(T) \right) = \frac{266}{225}.$$

Thus, we are done. □

(c) Suppose an agent sells the option in (ii) for V_0 at time zero. Compute the position Δ_0 so that at time one the value of her portfolio is V_1.

Solution. This problem is no different from what was done in Chapter 1. We find that

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{83}{270},$$

the same as always. □

(d) Suppose the first toss was heads. Find $\Delta_1(H)$ so that the agent's portfolio value at time two will be $(S_2 - 7)^+$.

Solution. Again, we approach this as we would in Chapter 1.

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{4}{4} = 1.$$

The problem is complete. □

2.10 Consider a dividend-paying stock (specifics of this model are in your text).

(a) Show the discounted wealth process is a martingale under the risk-neutral measure.

Proof. We start with a conditional expectation.

$$\begin{aligned}
\tilde{\mathbb{E}}_n \left[\frac{X_{n+1}}{(1+r)^{n+1}} \right] &= \frac{\Delta_n S_n \tilde{\mathbb{E}}_n Y_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}} \left[\frac{1+r-d}{u-d} u + \frac{u-1-r}{u-d} d \right] + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}} \cdot \frac{(u-d)(1+r)}{u-d} + \frac{X_n - \Delta_n S_n}{(1+r)^n} = \frac{X_n}{(1+r^n)}
\end{aligned}$$

Thus, the sequence is a martingale. □

(b) Show that the risk-neutral pricing formula still applies.

Proof. The risk-neutral pricing formula proof only depended on the fact that the discounted wealth process is a martingale. Since we have now shown that the discounted wealth process for dividend-paying stock is still a martingale, the previous proof (in text and exercise 2.8) holds. □

(c) Show that the discounted stock price is not a martingale under the risk-neutral measure. However, if A_{n+1} is a constant $a \in (0,1)$, regardless of the value of n and the outcome of the coin tosses, then $\frac{S_n}{(1-a)^n(1+r)^n}$ is a martingale under the risk-neutral measure.

Proof. We first show that the discounted stock price is not a martingale.

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{Y_{n+1}S_n - A_{n+1}Y_{n+1}S_n}{(1+r)^{n+1}}\right] \\
&= S_n\frac{\tilde{\mathbb{E}}_nY_{n+1} - \tilde{\mathbb{E}}_nA_{n+1}Y_{n+1}}{(1+r)^{n+1}} = \frac{S_n}{(1+r)^n}\left(\frac{1+r}{1+r} - \frac{\tilde{\mathbb{E}}_nA_{n+1}Y_{n+1}}{1+r}\right) \\
&= \frac{S_n}{(1+r)^n}\left(1 - \frac{\tilde{\mathbb{E}}_nA_{n+1}Y_{n+1}}{1+r}\right) \neq \frac{S_n}{(1+r)^n}
\end{aligned}$$

So the sequence is not a martingale. However,

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1-a)^{n+1}(1+r)^{n+1}}\right] &= \frac{S_n}{(1-a)^{n+1}(1+r)^n}\left(1 - \frac{\tilde{\mathbb{E}}_naY_{n+1}}{1+r}\right) \\
&= \frac{S_n}{(1-a)^{n+1}(1+r)^n}(1-a) = \frac{S_n}{(1-a)^n(1+r)^n}
\end{aligned}$$

illustrates that $\frac{S_n}{(1-a)^n(1+r)^n}$ is a martingale. □

2.11 Consider an N-period binomial model. A European call has payoff $C_N = (S_N - K)^+$, with the price C_n given by

$$C_n = \tilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right], \quad n = 0, 1, \ldots, N-1.$$

A put and forward contract are defined similarly.

(a) Explain why $C_N = F_N + P_N$.

Solution. If $S_N > K$, then $C_N = S_N - K = S_N - K + 0 = F_N + P_N$. On the other hand, if $S_N < K$ we get $C_N = 0 = S_N - K + K - S_N = F_N + P_N$. Finally, if $S_N = K$ then both sides are zero. □

(b) Show that $C_n = F_n + P_n$ for every n.

Proof. Since $C_N = F_N + P_N$, we have the following equalities:

$$\begin{aligned}
\frac{C_N}{(1+r)^{N-n}} &= \frac{F_N + P_N}{(1+r)^{N-n}} \\
\tilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] \\
\tilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \tilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right] \\
C_n &= F_n + P_n
\end{aligned}$$

which was to be proved. □

(c) Using the fact that the discounted stock price is a martingale under the risk-neutral measure, show that $F_0 = S_0 - \frac{K}{(1+r)^N}$.

Proof. We know that

$$F_0 = \tilde{\mathbb{E}}_0\left[\frac{S_N - K}{(1+r)^N}\right] = \tilde{\mathbb{E}}_0\left[\frac{S_N}{(1+r)^N}\right] - \frac{K}{(1+r)^N}$$
$$= S_0 - \frac{K}{(1+r)^N},$$

which was to be proved. □

(d) Suppose you begin at time zero wiht F_0, buy one share of stock, borrowing money as necessary to do that, and make no further trades. Show that at time N you have a portfolio valued at F_N.

Proof. At time N the portfolio value will be

$$S_N + (1+r)^N(F_0 - S_0) = S_N + (1+r)^N\left(S_0\frac{K}{(1+r)^N} - S_0\right) = S_N - K = F_N,$$

so the claim is true. □

(e) Show that, at time zero, the price of a call struck at the forward price is the same as the price of a put struck at the forward price.

Proof. From (b), we have $C_0 = F_0 + P_0$. But if the strike price is the forward price, then $F_0 = 0$. So $C_0 = P_0$. □

(f) If we choose $K = (1+r)^N S_0$, do we have $C_n = P_n$ for every n?

Solution. No. To see this, consider $n = N$. □

2.12 Show that the time zero price of a chooser option is the sum of a put, expiring at time N and having strike price K, and a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$.

Proof. We know that

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]$$

for every n. Now, $V_N = C_N$ if the buyer chooses the call at time m and $V_N = P_N$ if the buyer chooses the put. Thanks to our work on 2.11, we know that $C_N =$

$P_N + F_N$. So we can express V_N by $V_N = F_N + \delta_m P_N$ where δ_m is 0 or 1, depending on the buyer's choice at time m. From this we get

$$\begin{aligned}
V_0 = \tilde{\mathbb{E}}_0\left[\frac{P_N + \delta_m F_N}{(1+r)^N}\right] &= \tilde{\mathbb{E}}_0\left[\frac{P_N}{(1+r)^N}\right] + \tilde{\mathbb{E}}_0\left[\frac{\delta_m F_N}{(1+r)^N}\right] \\
&= P_0 + \tilde{\mathbb{E}}_0\left[\delta_m\left[\frac{F_N}{(1+r)^N}\right]\right] \\
&= P_0 + \tilde{\mathbb{E}}_0\tilde{\mathbb{E}}_m\left[\delta_m\left[\frac{F_N}{(1+r)^N}\right]\right] \quad \text{(by repeated conditioning)} \\
&= P_0 + \tilde{\mathbb{E}}_0\left[\delta_m\tilde{\mathbb{E}}_m\left[\left[\frac{F_N}{(1+r)^N}\right]\right]\right] \quad \text{(taking out what is known)} \\
&= P_0 + \tilde{\mathbb{E}}_0\left[\delta_m\frac{F_m}{(1+r)^m}\right] \quad \text{(martingale property).}
\end{aligned}$$

Just like 2.11 (c), we have

$$F_m = S_m - \frac{K}{(1+r)^{N-m}}.$$

(**Do you know how to prove this?**). Hence

$$V_0 \quad = \quad P_0 + \tilde{\mathbb{E}}_0\left[\delta_m\left(\frac{S_m}{(1+r)^m} - \frac{K}{(1+r)^N}\right)\right]. \qquad (1)$$

The optimal strategy δ_m that maximizes the value V_0 is: $\delta_m = 1$ when $\left(\frac{S_m}{(1+r)^m} - \frac{K}{(1+r)^N}\right) > 0$ and $\delta_m = 0$ otherwise.

So we get

$$\begin{aligned}
V_0 &= P_0 + \tilde{\mathbb{E}}_0\left[\left(\frac{S_m}{(1+r)^m} - \frac{K}{(1+r)^N}\right)^+\right] \\
&= P_0 + \tilde{\mathbb{E}}_0\left[\frac{C_m}{(1+r)^m}\right] = P_0 + C_0
\end{aligned}$$

where

$$C_m = (S_m - \frac{K}{(1+r)^{N-m}})^+$$

denotes the value of the call expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$. $\square$

2.13 Consider an N=period binomial model. An *Asian option* has a payoff based on the average stock price, i.e.,

$$V_N = A\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right),$$

where the function A is determined by the contractual details of the option.

(a) Define $Y_n = \sum_{k=0}^{n} S_k$ and show that (S_n, Y_n) is a Markov process.

Proof. Let $Z_{n+1} = S_{n+1}/S_n$. So this random variable is solely dependent on the $n+1$ coin toss. Then we can make the following expression

$$\mathbb{E}_n[f(S_{n+1}, Y_{n+1})] = \mathbb{E}_n[f(S_n Z, Y_n + S_n Z)].$$

Following example 2.5.6 and 2.5.7, we set

$$g(s,y) = \mathbb{E}f(sZ, y + sZ) = pf(su, y+su) + qf(sd, y+sd).$$

By Lemma 2.3.5, we get $\mathbb{E}_n[f(S_{n+1}, Y_{n+1})] = g(S_n, Y_n)$. Thus, the chain is Markov. □

(b) Give a formula for $v_N(s,y)$, and provide an algorithm for computing $v_n(s,y)$ in terms of v_{n+1}.

Solution. First, it is clear that we should define $v_N(s,y) = A(Y/(N+1))$. Next, we suppose we have a formula for $n+1$. Then we get

$$\begin{aligned} V_n &= \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{1+r}\right] = \frac{\tilde{\mathbb{E}}_n[v_{n+1}(S_{n+1}, Y_{n+1})]}{1+r} \\ &= \frac{g(S_n, Y_n)}{1+r} \end{aligned}$$

where this is the same g from part (a). Thus, we should clearly define $v_n(s,y)$ by

$$v_n(s,y) = \frac{1}{1+r}\left[\tilde{p}v_{n+1}(su, y+su) + \tilde{q}v_{n+1}(sd, y+sd)\right].$$

The problem is complete. □

2.14 Consider an N=period binomial model. An *Asian option* has a payoff based on the average stock price, i.e.,

$$V_N = A\left(\frac{1}{N-M}\sum_{n=M+1}^{N} S_n\right),$$

where the function A is determined by the contractual details of the option.

(a) Define $Y_n = \sum_{k=M+1}^{n} S_k$ for $n \geq M+1$ and 0 otherwise and show that (S_n, Y_n) is a Markov process.

Proof. This procedure is very similar to that of 2.13. We define Z_{n+1} as before. Note that $Y_{n+1} = Y_n + ZS_n$ for $n = M, \ldots, N$ and is 0 otherwise. Then we have

$$\tilde{\mathbb{E}}_n[f(S_{n+1}, Y_{n+1})] = \tilde{p}f(uS_n, Y_n + uS_n) + \tilde{q}f(dS_n, Y_n + dS_n)$$

for $n = M, \ldots, N$. For $n < M$, we get

$$\tilde{\mathbb{E}}_n[f(S_{n+1}, Y_{n+1})] = \tilde{p}f(uS_n, 0) + \tilde{q}f(dS_n, 0).$$

So we define $g_n(s,y) = \tilde{p}f(us, y+us) + \tilde{q}f(ds, y+ds)$ for $n = M, \ldots, N$ and $g_n(s,y) = \tilde{p}f(us, 0) + \tilde{q}f(ds, 0)$ for all other n. □

(b) Give a formula for $v_N(s, y)$, and provide an algorithm for computing $v_n(s, y)$ in terms of v_{n+1}.

Solution. The first clear definition is $v_N(s, y) = A(Y_N/(N - M))$. Then, supposing we have a formula for v_{n+1}, we note that

$$V_n = \frac{\mathbb{E}_n V_{n+1}}{1+r} = \frac{\mathbb{E}_n v_{n+1}(S_{n+1}, Y_{n+1})}{1+r} = \frac{g_n(S_n, Y_n)}{1+r}$$

where g_n is from (a), but now it is dependent on v_{n+1} rather than f. Therefore, we should define

$$v_n(s, y) = \frac{g_n(s, y)}{1+r},$$

keeping in mind that each g_n is a function of v_{n+1}. Note that for $n \leq M$, v_n is only a function of s (look at definition of g_n for explanation). Thus, the algorithm is complete. $\square$