Math 133: Homework 6

Prepared by Gregory Shinault
gshinault@math.ucdavis.edu

1. Two stocks are available. The corresponding expected rates of return are $\bar{r}_1, \bar{r}_2$; the corresponding variances and covariances are $\sigma_1^2, \sigma_2^2, \sigma_{12}$. What percentages of total investent should be invested in each of the two stocks to minimize the total variance of the rate of return of the resulting portfolio? What is the mean rate of return of this portfolio?

   Solution. We want to minimize $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$ subject to the constraint $w_1 + w_2 = 1$. We know $w_2 = 1 - w_1$, so

   $$\sigma_p^2 = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{12}.$$

   To minimize this variance, we apply the following method:

   $$\partial_{w_1}\sigma_p^2 = 2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2 + 2\sigma_{12} - 4w_1\sigma_{12} = 0.$$

   Solving for w_1, we get

   $$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \quad w_2 = 1 - w_1 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

   For the expected return, we get

   $$\mathbb{E}[r] = w_1\bar{r}_1 + w_2\bar{r}_2 = \frac{\bar{r}_1\sigma_2^2 + \bar{r}_2\sigma_1^2 - \sigma_{12}(\bar{r}_1 + \bar{r}_2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

   by directly applying what was computed in the first part of the problem. □

2. There are just three assets with rates of return r_1, r_2, and r_3, respectively. The covariance matrix and the expected rate of return are

   $$\mathbf{C} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \bar{\mathbf{r}} = \begin{bmatrix} 0.4 \\ 0.8 \\ 0.8 \end{bmatrix}.$$

   (a) Find the minimum variance portfolio.

   Solution. First note that $w_1 = w_3$ by symmetry. Our goal is to minimize the variance $\sigma^2 = \Sigma_{i,j} w_i w_j \sigma_{ij}$ subject to the constraint $\sigma_{j=1}^3 w_j = 1$. We apply the

Lagrange method, with Lagrangian

$$\begin{aligned}\mathcal{L} &= \sigma^2 - \mu\left(\sum_{j=1}^{3} w_j - 1\right)\\ &= (2w_1^2 + 2w_1w_2 + 2w_2^2 + 2w_2w_3 + 2w_3^2) - \mu\left(\sum_{j=1}^{3} w_j - 1\right)\\ &= (4w_1^2 + 4w_1w_2 + 2w_2^2) - \mu(2w_1 + w_2 - 1).\end{aligned}$$

Taking partial derivatives, we get the linear system of equations

$$\begin{aligned}\partial_{w_1}\mathcal{L} &= 8w_1 + 4w_2 - 2\mu = 0\\ \partial_{w_2}\mathcal{L} &= 4w_1 + 4w_2 - \mu = 0.\end{aligned}$$

We first solve for w_1 and find that $w_1 = \mu/4$. Substituting this into either equation we will find $w_2 = 0$. Since $w_1 + w_2 + w_3 = 1$ and $w_1 = w_3$, we get $w_1 = w_3 = 1/2$. □

(b) Find another efficient portfolio by setting $\lambda = 1, \mu = 0$.

Solution. We must minimize σ^2 subject to the constraints $\Sigma w_j = 1$ and $\Sigma w_j \bar{r}_j = \bar{r}$. The Lagrangian for a problem with two constraints is

$$\mathcal{L} = \sigma^2 - \mu\left(\Sigma w_i - 1\right) - \lambda\left(\Sigma w_i \bar{r}_i - \bar{r}\right)$$

where this time we take $\lambda = 1, \mu = 0$. So

$$\mathcal{L} = \sigma^2 - \left(\Sigma w_i \bar{r}_i - \bar{r}\right).$$

This means the constraint $\Sigma w_j = 1$ is not imposed. So we have to normalize the weights derived subsequently.

We do not expect symmetry in this case and we proceed directly.

$$\begin{aligned}\partial_{w_1}\mathcal{L} &= 4w_1 + 2w_2 - \bar{r}_1 = 0\\ \partial_{w_2}\mathcal{L} &= 2w_1 + 4w_2 + 2w_3 - \bar{r}_2 = 0\\ \partial_{w_3}\mathcal{L} &= 2w_2 + 4w_3 - \bar{r}_3 = 0.\end{aligned}$$

We can rewrite this in matrix form:

$$\begin{bmatrix} 4 & 2 & 0\\ 2 & 4 & 2\\ 0 & 2 & 4\end{bmatrix}\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix} = \begin{bmatrix}\bar{r}_1\\ \bar{r}_2\\ \bar{r}_3\end{bmatrix}.$$

This is easily solved with linear algebra,

$$\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix} = \begin{bmatrix} 4 & 2 & 0\\ 2 & 4 & 2\\ 0 & 2 & 4\end{bmatrix}^{-1}\begin{bmatrix}\bar{r}_1\\ \bar{r}_2\\ \bar{r}_3\end{bmatrix}.$$

Once this is solved, we normalize the weights by $w_i/\sum_j w_j, i = 1, .., n$. I leave the final computation to the reader. □

(c) If the risk-free rate $r_f = 0.2$, find the efficient portfolio (i.e. *tangent* fund) of risky assets.

Solution. The tangency condition is

$$\lambda \mathbf{C}\mathbf{w} = (\bar{r}_1 - r_f, \bar{r}_2 - r_f, \bar{r}_3 - r_f)^T = (0.2, 0.6, 0.6)^T$$

Let $\mathbf{v} = \lambda \mathbf{w}$. Hence

$$\mathbf{v} = \mathbf{C}^{-1}(0.2, 0.6, 0.6)^T$$

and we can express the solution as

$$w_i = \frac{v_i}{\sum_j v_j}, \qquad i = 1, 2, 3.$$

□

3. Suppose there are n assets which are uncorrelated. You may invest in any one or in any combination of them. The mean rate of return $\bar{r}$ is the same for each asset, but the variances are different. The return on asset i has a variance of σ_i^2.

(a) Describe the feasible set and the efficient frontier on the $\bar{r}$-σ plane.

Solution. The feasible set is just a horizontal line and the point with the lowest variance is the efficient frontier. □

(b) Find the minimum-variance point. Express your result in terms of

$$\bar{\sigma}^2 = \left(\sum_{i=1}^{n} \sigma_i^{-2} \right)^{-1}.$$

Solution. Let's minimize the variance

$$\sigma^2 = \sum_{i,j} w_i w_j \sigma_{ij}$$

subject to the constraint $\sum w_i = 1$ as usual. This leads to

$$\frac{\partial L}{\partial w_i} = 2 w_i \sigma_{ii} - \mu = 0, \quad \forall i = 1, ..., n$$

and hence

$$w_i = \frac{\mu}{2\sigma_{ii}}.$$

Using this in the constraint we deduce that

$$\mu = 2\bar{\sigma}^2.$$

Therefore $w_i = \bar{\sigma}/\sigma_{ii}$. □

(c) Suppose the total amount of asset i in the market is X_i. Let $T = \sum_{i=1}^n$ and set $x_i = X_i/T$. x_i is the market share of asset i. We can think of the **market portfolio** in the normalized form as $\mathbf{x} = (x_1, \ldots, x_n)$. Let r_f be the risk-free rate. Find an expression for β_i in terms of the x_i's and σ_i's.

Solution. We need to rewrite

$$\beta_i = \frac{\mathrm{Cov}(r_i, \sum_j x_j r_j)}{\mathrm{Var}(\sum_j x_j r_j)}, \quad i = 1, ..., n$$

in terms of x_i, σ_i. Consider the following calculations

$$\mathbb{E}(\sum_j x_j(r_j - \bar{r}))^2 = \mathbb{E}(\sum_{j,k} x_j x_k (r_j - \bar{r})(r_k - \bar{r})) = \sum_{j,k} x_j x_k \sigma_{jk}$$

and

$$\mathbb{E}(r_i - \bar{r})(\sum_j x_j r_J - \bar{r}) = \sum_j x_j \sigma_{ij}.$$

Therefore

$$\beta_i = \frac{\sum_j \sigma_{ij} x_j}{\sum_{j,k} \sigma_{jk} x_j x_k}$$

□

4. Assume that the expected rate of return on the market portfolio is 23% and the rate of return on T=bills (the risk-free rate) is 7%. The standard deviation of the market is 32%. Assume that the market portfolio is efficient.

(a) What is the equation of the market line?

Solution. The equation for the market line is

$$\bar{r} = r_f + \frac{(r_M - r_f)\sigma_p}{\sigma_M} = .07 + .5\sigma.$$

□

(b) If an expected return of 39% is desired, what is the standard deviation of this position?

Solution. Proceed as follows:

$$.39 = .07 + .5\sigma \;\Rightarrow\; \sigma = .64.$$

This is just a substitution into our solution from part (a). □

(c) If you have \$1000 to invest, how should you allocate it to achieve the above postion?

Solution. We know that we want $w_1\bar{r}_1 + w_2\bar{r}_2 = .39$. That is

$$w_1(.07) + (1 - w_1)(.23) = .39.$$

This leads to the equation $w_1(.07 - .23) = .39 - .23$. The punch line is that $w_1 = -1$ and $w_2 = 2$. So to achieve this postion, we must actually borrow. □

(d) If you invest \$300 in the risk-free asset and \$700 in the market portfolio, how much money should you expect to have at the end of the year?

Solution. You should expect \$300(1.07) + \$700(7.23) = \$1182. □

5. Consider a world with only two risky assets and a risk-free asset. The two risk assets are in equal supply in the market. The following information is known: $r_f = 0.1, \sigma_1^2 = 0.04, \sigma_{12} = 0.01, \sigma_2^2 = 0.02, \bar{r}_M = 0.18$.

(a) Find a general expression for σ_M^2, β_1, and β_2.

Solution. First σ_M^2:

$$\begin{aligned}\sigma_M^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}\\ &= (.5)^2(.04) + (.05)^2(.02) + 2(.05)(.05)(.01)\\ &= .01 + .005 + .005 = .02\end{aligned}$$

Next, β_1:

$$\beta_1 = \frac{2\left[\sigma_1^2 + \sigma_{12}\right]}{\sigma_1^2 + 2\sigma_{12} + \sigma_2^2}$$

β_2 is determined similarly. □

(b) According to CAPM, what are the numerical values of $\bar{r}_1, \bar{r}_2$?

Solution. According to CAPM, $\bar{r}_j = r_f + \beta_j(r_M - r_f)$. So,

$$\bar{r}_1 = .10 + \beta_1(.18 - .1) = .10 + 1.25(.08) = .20$$

$$\bar{r}_2 = .10 + \beta_2(.18 - .1) = .10 + 0.75(.08) = .16$$

□