HW7-solutions

(1) Compute the state prices for the Film Venture example discussed in the class.

Solution There are two securities: film venture and risk-free asset. Let d_1 and d_2 be their respective dividends. Let w_1 and w_2 be the weights invested in the two securities. $w_1 + w_2 = 1$.

Let P_1 and P_2 be the price per unit share of the two securities respectively. Let θ_1 and θ_2 be the number of shares purchased for the two securities. Hence

$$\theta_i P_i = w_i W, \quad i = 1, 2$$

and the total payoff is

$$x = \sum_j \theta_i d_i.$$

For this problem it is assumed that $P_1 = P_2 = 1$. Also, $d_1 = 3$ when highly successful, $d_1 = 1$ when moderately successful and $d_1 = 0$ when failing. $d_2 = 1.2$ in all three states of affair.

The optimal portfolio corresponds to maximizing the Lagrangian

$$L = \mathbb{E}(U(x)) - \lambda(\sum_j \theta_j P_j - W)$$

where $\lambda > 0$ is the Lagrangian multiplier. Differentiating L with θ_i and setting it to zero we find the necessary condition

$$\mathbb{E}[U'(x_*)d_i] = \lambda P_i, \quad i = 1, 2$$

where x_* is the (random) payoff of the optimal portfolio. If we expand the equation to show the details, we find

$$P_i = \frac{1}{\lambda} \sum_{j=1}^{s} p_j U'(x_*)\big|_{S_j} d_i^{(j)}$$

where $d_i^{(j)}$ is the dividend (payoff) in the case of the $j-$th state S_j and p_j is the probability of its occurrences. Now the state price for S_j is defined as

$$\psi_j = \frac{p_j U'(x_*)\big|_{S_j}}{\lambda}$$

in terms of which we can write

$$P_i = \sum_{j=1}^{s} \psi_j d_i^{(j)}.$$

We have found in the class that the optimal portfolio is

$$\theta_1 = 0.089W, \quad \theta_2 = 0.911W, \quad \lambda = 1/W$$

where W is the initial wealth. Therefore, we obtain that for the log-utility function

$$\begin{aligned}\psi_1 &= \frac{0.3}{3\theta_1 + 1.2\theta_2} = 0.221\\ \psi_2 &= \frac{0.4}{\theta_1 + 1.2\theta} = 0.338\\ \psi_3 &= \frac{0.3}{1.2\theta_2} = 0.274\end{aligned}$$

□

(2) At the horse races one Saturday afternoon Eddy studies the racing form and concludes that the horse "Wind" has a 25% chance to win and is posted at 4 to 1 odds. (For every dollar Eddy bets, he receives \$5 if the horse wins and nothing if it loses.) He can either bet on this horse or keep his money in his pocket. Eddy decides that he has a log utility for his money.

(a) What fraction of his money should Eddy bet on Wind? *Solution.* Let θ_1 be the dollar amount betted and θ_2 the dollar amount withheld in the pocket. We have $\theta_1 + \theta_2 = W$ where W is the initial wealth. Let d_1 be the dividend per dollar betted so $d_1 = 5$ when winning and $d_1 = 0$ when losing. And d_2 is the dividend of every dollar in the pocket so $d_2 = 1$, winning or losing.

Maximizing the Lagrangian

$$L = \mathbb{E}(U(x)) - \lambda(\theta_1 + \theta_2 - W)$$

we find the necessary condition:

$$\mathbb{E}[U'(x_*)d_i] = \lambda, \quad i = 1, 2.$$

With $U(x) = \ln x$, this becomes

$$\begin{aligned}\frac{0.25 \times 5}{5\theta_1 + \theta_2} &= \lambda\\ \frac{0.25}{5\theta_1 + \theta_2} + \frac{0.75}{\theta_2} &= \lambda.\end{aligned}$$

Solve for $\theta_1, \theta_2, \lambda$ from the above system and the constraint $\theta_1 + \theta_2 = W$. This gives the optimal strategy for allocation. □

(b) What are the state prices for the bet?
Solution. Compute

$$\psi_i = \frac{p_i U'(x_*)\big|_{S_i}}{\lambda}, \quad i = 1, 2$$

with $U(x) = \ln x$ as in Ex #1. □

(3) Consider the log-optimal pricing formula:

$$P = \mathbf{E}\left(\frac{d}{1 + r_*}\right) = \sum_{i=1}^{s} \frac{p_i d^{(i)}}{1 + r_*^{(i)}}$$

where $r_*^{(i)}$ is the log-optimal return rate r_* in the case of state i and the risk-neutral pricing formula

$$P = \widetilde{\mathbf{E}}(\frac{d}{1+r_f}) = \sum_{i=1}^{s} \frac{\tilde{p}_i d^{(i)}}{1+r_f}.$$

Express $r_* = (r_*^{(1)}, ..., r_*^{(s)})$ in terms of the risk-neutral probabilities $\tilde{p}_i$.

Solution. Comparing the two formulas for P we obtain

$$\frac{\tilde{p}_i d^{(i)}}{1+r_f} = \frac{p_i d^{(i)}}{1+r_*^{(i)}}$$

or equivalently

$$r_*^{(i)} = \frac{p_i - \tilde{p}_i + p_i r_f}{\tilde{p}_i}.$$

□