Math 133: Homework 1

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1. Consider an investment opportunity that generates the following stream of cash flow:

$$-1000, -1200, 800, 900, 800.$$

Suppose you can **borrow and/or lend** money at the interest rate 6% at will. Will you go for the above investment opportunity?

Solution. To proceed, we compute the present value of this opportunity:

$$\begin{aligned}\mathbf{PV} &= \sum_{t=0}^{4} \frac{C_t}{(1+r)^t} = -1000 + \frac{-1200}{1.06} + \frac{800}{1.06^2} + \frac{900}{1.06^3} + \frac{800}{1.06^4} \\ &\approx -1000.00 - 1132.08 + 712.00 + 755.66 + 633.67 = -30.75\end{aligned}$$

Since we have negative present value, this is not a worthwhile investment. □

2. Let P_j be the principal reduction of a mortgage (loan amount L with monthly interest rate r) in the j-th payment. Show that

$$\begin{aligned}P_j &= \frac{Lr(1+r)^{j-1}}{(1+r)^n - 1} \\ L &= \sum_{j=1}^{n} P_j\end{aligned}$$

where n is the total number of payments.

Proof. To obtain the first equality, we proceed in steps. First we will obtain the relation $P_j = P_1(1+r)^{j-1}$ for $j \geq 2$. To do this, we use induction. Our base case, $j = 2$, is fairly simple. Since $A = P_j + I_j$ for all j, $P_1 + I_1 = P_2 + I_2$. Then $P_2 = P_1 + I_1 - I_2$. Now, $I_1 = Lr$ and $I_2 = (L - P_1)r$, so

$$P_2 = P_1 + Lr - (L - P_1)r = P_1(1+r)^{2-1}$$

So our base case holds true. We know assume the relation holds for some $k-1$ and show that it will hold for k. We use the identity $P_k + I_k = P_{k-1} + I_{k-1}$ and the fact $I_j = (L - \sum_{i=1}^{j-1} P_i)r$.

$$\begin{aligned}P_k &= P_{k-1} + I_{k-1} + I_k = P_{k-1} + \left(\sum_{i=1}^{k-2} P_i\right) r - \left(\sum_{i=1}^{k-1} P_i\right) r \\ &= P_{k-1} + P_{k-1}r = P_{k-1}(1+r) \\ &= P_1(1+r)^{k-2}(1+r) = P_1(1+r)^{k-1}\end{aligned}$$

Thus, the relation holds for k. By the principle of mathematical induction, this is true for all $j \geq 2$.

We now use this fact and the relation $L = \sum_{j=1}^{n} P_j = P_1 \sum_{j=1}^{n} (1+r)^{j-1}$ to find P_1, which will finally give us the first equality. Note that this is not circular reasoning, though the second part of the problem asks us to prove $L = \sum_{j=1}^{n} P_j$. Part 2 is asking us to prove equality for the specific P_j determined in the first part, a sort of check to verify that the formula we determine for P_j is correct. Therefore,

$$P_1 = L\left(\sum_{j=0}^{n-1}(1+r)^j\right)^{-1} = L\left(\frac{(1+r)^n - 1}{(1+r)-1}\right)^{-1} = \frac{Lr}{(1+r)^n - 1}.$$

This fact, together with $P_j = P_1(1+r)^{j-1}$ yields

$$P_j = \frac{Lr}{(1+r)^n - 1}(1+r)^{j-1} = \frac{Lr(1+r)^{j-1}}{(1+r)^n - 1}.$$

The second equality follows from the formula for geometric sums.

$$\sum_{j=1}^{n} P_j = \frac{Lr}{(1+r)^n - 1}\sum_{j=0}^{n-1}(1+r)^j = \frac{Lr}{(1+r)^n - 1}\frac{(1+r)^n - 1}{(1+r)-1} = L$$

Thus, we are done. □

3. Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two streams of cash flow. Let

$$X_j = \sum_{i=1}^{j} x_i, \quad \bar{X}_j = \sum_{i=1}^{j} X_i,$$

$$Y_j = \sum_{i=1}^{j} y_i, \quad \bar{Y}_j = \sum_{i=1}^{j} Y_i,$$

for $j = 1, \ldots, n$. Suppose the interest rate r is nonnegative. Prove that if $X_n \geq Y_n$ and $\bar{X}_j \geq \bar{Y}_j$, for $j = 1, \ldots, n$ then the first stream is more profitable (i.e. has larger present value).

Proof. Our goal here is to show that $\mathbf{PV}_x \geq \mathbf{PV}_y$. So we start we $\mathbf{PV}_x$ and define

$X_0 = 0$ for convenience.

$$\mathbf{PV}_x = \sum_{j=1}^{n} \frac{x_j}{(1+r)^j} = \sum_{j=1}^{n} \frac{X_j - X_{j-1}}{(1+r)^j} = \sum_{j=1}^{n} \frac{X_j}{(1+r)^j} - \sum_{j=1}^{n} \frac{X_{j-1}}{(1+r)^j} \tag{1}$$

$$= \sum_{j=1}^{n} \frac{X_j}{(1+r)^j} - \sum_{j=1}^{n-1} \frac{X_j}{(1+r)^{j+1}} \tag{2}$$

$$= \frac{X_n}{(1+r)^n} + \sum_{j=1}^{n-1} \frac{X_j}{(1+r)^j} - \frac{1}{1+r} \sum_{j=1}^{n-1} \frac{X_j}{(1+r)^j} \tag{3}$$

$$= \frac{X_n}{(1+r)^n} + \left(1 - \frac{1}{1+r}\right) \sum_{j=1}^{n-1} \frac{X_j}{(1+r)^j} \tag{4}$$

$$= \frac{X_n}{(1+r)^n} + \frac{r}{1+r} \sum_{j=1}^{n-1} \frac{X_j}{(1+r)^j} \tag{5}$$

Using this same argument, we can obtain

$$\sum_{j=1}^{n} \frac{X_j}{(1+r)^j} = \frac{\bar{X}_n}{(1+r)^n} + \frac{r}{1+r} \sum_{j=1}^{n-1} \frac{\bar{X}_j}{(1+r)^j}$$

which is substituted into (5) and we have

$$\mathbf{PV}_x = \frac{X_n}{(1+r)^n} + \frac{r}{1+r} \left(\frac{\bar{X}_{n-1}}{(1+r)^{n-1}} + \frac{r}{1+r} \sum_{j=1}^{n-2} \frac{\bar{X}_j}{(1+r)^j} \right)$$

Similarly,

$$\mathbf{PV}_y = \frac{Y_n}{(1+r)^n} + \frac{r}{1+r} \left(\frac{\bar{Y}_{n-1}}{(1+r)^{n-1}} + \frac{r}{1+r} \sum_{j=1}^{n-2} \frac{\bar{Y}_j}{(1+r)^j} \right)$$

and $X_n \geq Y_n$, $\bar{X}_{n-1} \geq \bar{Y}_{n-1}$, $\sum_{j=1}^{n-2} \frac{\bar{X}_j}{(1+r)^j} \geq \sum_{j=1}^{n-2} \frac{\bar{Y}_j}{(1+r)^j}$, and $r > 0$ give our desired result: $\mathbf{PV}_x \geq \mathbf{PV}_y$. □

4. Suppose you can borrow money at an annual rate of 8% but can save money at an annual rate of only 5%. Consider and investment opportunity with the following cash flow (the initial captial/payment is included):

$$-1000, 900, 800, -1200, 700.$$

Should you invest?

Solution.

Let's consider the following scenario accounting for both the investment and saving/lending: starting with zero capital, borrow 1000 from the bank to make the first payment, pay back 900 and 800 with the second and third cash flows of the investment, borrow more money from the bank to make the payment 1200 and finally repay the debt using last incoming 700.

The value of the combined investment at the end of the fourth period is given by

$$\begin{aligned} & \{[(-1000 \times 1.08 + 900) \times 1.08 + 800] \times 1.05 - 1200\} \times 1.08 + 700 \\ = & -1000 \times (1.08)^3 \times 1.05 + 900 \times (1.08)^2 \times 1.05 + 800 \times 1.08 \times 1.05 - 1200 \times 1.08 + 700 \\ = & 90.7504 \end{aligned}$$

which is positive. Therefore, the investment is profitable. □

5. Consider two cash flow streams:

$$100, 140, 131 \text{ and } 90, 160, 120.$$

Is it possible to tell which cash flow stream is preferable without knowing the iterest rate?

Solution. It is possible and the first cash flow stream is more profitable. To see this, let $\mathbf{PV}_1$ and $\mathbf{PV}_2$ denote the present values of the first and second cash flow streams, respectively. In fact, we can view these terms as a function of u, where $u = 1 + r$.

$$\mathbf{PV}_1(u) = 100 + \frac{140}{u} + \frac{131}{u^2}$$
$$\mathbf{PV}_2(u) = 90 + \frac{160}{u} + \frac{120}{u^2}$$

If we show that $\mathbf{PV}_1(u) - \mathbf{PV}_2(u)$ is either always negative or always positive, then we can determine which stream is preferable. Since $\mathbf{PV}_1(u) - \mathbf{PV}_2(u)$ is a continuous function (because we restrict r to be nonnegative, so $u \geq 1$), the intermediate value theorem tells us that if the function attains both positive and negative values then it must equal zero somewhere. So we look for this value. We get

$$\mathbf{PV}_1(u) - \mathbf{PV}_2(u) = 10 - \frac{20}{u} + \frac{11}{u^2} = 0$$

which is equivalent to

$$10u^2 - 20u + 11 = 0.$$

Applying the quadratic formula, we find that the roots of this polynomial are complex. Since $\mathbf{PV}_1(1) - \mathbf{PV}_2(1) > 0$, $\mathbf{PV}_1(u) - \mathbf{PV}_2(u) > 0$ for all values of $u \geq 1$.

A better method might be applying our work from 3. We will get

$$X_1 = 100, X_2 = 240, X_3 = 371 \;\&\; Y_1 = 90, Y_2 = 250, Y_3 = 370$$

and

$$\bar{X}_1 = 100, \bar{X}_2 = 340, \bar{X}_3 = 711 \;\&\; \bar{Y}_1 = 90, \bar{Y}_2 = 340, \bar{Y}_3 = 710.$$

Since $\bar{X}_i \geq \bar{Y}_i$ for $i = 1, 2, 3$ and $X_3 \geq Y_3$, Exercise 3 tells us that the first stream is more profitable. □