Math 133: Homework 2

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For all of the following problems, we consider an investment of period T.

1. Suppose the delivery price K of a forward contract is **not** exactly the T-forward price, i.e.,

$$K \neq S_0 e^{rT} \equiv K_T.$$

Show that the fair price/value of this forward contract is

$$S_0 - Ke^{-rT}$$

for the long position.

Proof. Consider two investments A and B. In A you buy a share of stock at the price S_0 and hold. In B, you spend F (the price of the long forward) to enter the long position of the forward contract with delivery price K and put cash Ke^{-rT} in the money market.

At T, investment A produces one share of stock and investment B also produces one share of stock by taking out the savings K and use the forward to purchase a share of stock. Since A and B produce the identical result, they must have the same cost initially.

Since initially A costs S_0 and B costs $F + Ke^{-rT}$, we obtain

$$S_0 = F + K^{-rT}.$$

□

2. Let C and P be the prices of the call and put options, resprectively. Suppose the strike price K of the call and put options are **exactly** the same as the forward price of the underlying security. Show that $C = P$ following the discussion from Friday's lecture.

Proof. Recall the forward price is the price at which the forward contract requires no cost to enter either in the long or short position. Let C_T, P_T, F_T be the values of the call, put and the forward at expiration T. Since C_T, P_T, F_T satisfy the relation $C_T = P_T + F_T$, the portfolio holding one unit of call, negative one unit of put and negative one unit of forward must be worth zero initially, i.e.

$$C = P + F.$$

Since $F = 0$ (a forward contract with delivery price exactly equal to the forward price costs nothing to enter into), we deduce that $C = P$.

$\square$

3. Suppose the strike price K of the call and put options are **not** the same as the T-forward price of the underlying security. Show that

$$C + Ke^{-rT} = P + S_0$$

by analyzing the value of the portfolio and using (1).

Proof. The portfolio value for a call less a put option must give $C - P = F$. This then gives $C - P = S_0 - Ke^{-rT}$ by (1), so we can rearrange this equality and arrive at the correct conclusion, $C + Ke^{-rT} = P + S_0$. $\square$

4. Let K_T be the T-forward price of one euro in the unit of USD. Let S_0 be the spot price of the euro. Suppse r_u and r_e are the interest rates earned in the euro and the dollar, respectively. Prove that

$$K_T = S_0 e^{(r_e - r_u)T}.$$

Proof. We will consider two portfolios. The first investment is holding $K_T e^{-r_u T}$ USD in an American bank and entering the long forward. At time T, the cash holding is K_T USD just enough to allows us to buy one euro using the long forward.

The second portfolio holds $e^{-r_e T}$ euro in an European bank. At time T, this portfolio produces exactly one euro.

The two investments produce the identical result (one euro) at time T. Therefore, their initial values must be the same

$$S_0 e^{-r_e T} = K_T e^{-r_u T}$$

which implies $K_T = S_0 e^{(r_u - r_e)T}$. $\square$

5. A call and put option on the same stock both expire in three months, both have a strike price of 20 and both sell for the price of 3. If the nominal continuously compounded anual interest rate is 10% and the stock price is currently 25, identify an arbitrage.

Solution. Note that the identity established in (3), $C - P = S_0 - Ke^{-rT}$, for no arbitrage. If this identity fails then we know that an arbitrage exists. Indeed, we have

$$C - P = 0 \neq 25 - 20e^{-.10/4} = S_0 - Ke^{-rT}$$

so an arbitrage must exist (we know $C - P = 0$ from $C = P = 3$). With this in mind, we establish how to exploit the arbitrage.

Borrow 3 at the continuously compounded anual interest rate of 10%. Buy the call option. Shortsell a stock and invest that 25 at the risk-free rate. Three months later purchase the stock at strike price 20 and pay the bank $3e^{.10/4} \approx 3.08$. You should have $25e^{.10/4} - 20 - 3.08 \approx 2.55$ in your pocket. $\square$