Math 133: Homework 3

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1.2 We proceed by computing X_1 directly for $\omega_1 = H$ and $\omega_1 = T$.

$$\begin{aligned} X_1(H) &= \Delta_0 S_1(H) + \Gamma_0 (S_1 - 5)^+(H) - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) \\ &= 8\Delta_0 + 3\Gamma_0 - 5\Delta_0 1 - \frac{3}{2}\Gamma_0 = 3\Delta_0 + \frac{3}{2}\Gamma_0 \end{aligned}$$

and

$$\begin{aligned} X_1(T) &= \Delta_0 S_1(T) + \Gamma_0 (S_1 - 5)^+(T) - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) \\ &= 2\Delta_0 + 0\Gamma_0 - 5\Delta_0 1 - \frac{3}{2}\Gamma_0 = -3\Delta_0 - \frac{3}{2}\Gamma_0 \end{aligned}$$

so $X_1(H) = -X_1(T)$. Thus if one of them is positive, the other must be negative. Also, both H and T are assumed to have positive probability of occurring. So there is positive probability X_1 is negative.

1.3 Directly computing V_0 from (1.1.10) we get

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] = S_0.$$

The final equality comes from (1.1.6). Note this should be true if the law of one price is.

1.5 We first compute $X_1(H)$ using the wealth equation.

$$\begin{aligned} X_1(H) &= \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) \\ &= 8 \cdot .1733 + \frac{5}{4}(1.376 - .1733 \cdot 4) = 2.2399 \approx 2.24 \end{aligned}$$

The agent now takes the position $\Delta_1(H) = (3.20 - 2.40)/(16 - 4) = 1/15$. We proceed by computing the portfolio at time 2 for H and T.

$$\begin{aligned} X_2(HT) &= \Delta_1 S_2(HT) + (1+r)(X_1 - \Delta_1 S_1(H)) \\ &= 4 \cdot \frac{1}{15} + \frac{5}{4}(2.24 - \frac{1}{15} \cdot 8) = 2.40 \end{aligned}$$

and

$$\begin{aligned} X_2(HH) &= \Delta_1 S_2(HH) + (1+r)(X_1 - \Delta_1 S_1(H)) \\ &= 16 \cdot \frac{1}{15} + \frac{5}{4}(2.24 - \frac{1}{15} \cdot 8) = 3.20 \end{aligned}$$

so $X_2(HT) = V_2(HT)$ and $X_2(HH) = V_2(HH)$. We now assume the second toss was tails and the agent takes the position $\Delta_2(HT) = (0-6)/(8-2) = -$. We finally compute the portfolio at time 3 under these assumptions.

$$\begin{aligned} X_3(HTH) &= \Delta_2 S_3(HTH) + (1+r)(X_2 - \Delta_2 S_2(HT)) \\ &= -1\cdot 8 + \frac{5}{4}(2.40 - (-1)4) = 0 \end{aligned}$$

and

$$\begin{aligned} X_3(HTT) &= \Delta_2 S_3(HTT) + (1+r)(X_2 - \Delta_2 S_2(HT)) \\ &= -1\cdot 2 + \frac{5}{4}(2.40 - (-1)4) = 6 \end{aligned}$$

so $X_3(HTH) = V_3(HTH)$ and $X_3(HTT) = V_3(HTT)$.

1.6 Our goal is to create a portfolio such that $Y_1 = X_1 + V_1 = 1.50$, where V is the value of the bank's call and X is some stock and money market portfolio we are to determine. We know we want $Y_1(H) = X_1(H) + V_1(H) = X_1(H) + 3 = 1.50$ and $Y_1(T) = X_1(T) + V_1(T) = X_1(T) + 0 = 1.50$ which means that we require $X_1(H) = -1.50$ and $X_1(T) = 1.50$. So we must now determine how to build our X portfolio. Applying the delta hedging formula to our desired portfolio values, we obtain $\Delta_0 = (-1.50 - 1.50)/(8-2) = (V_1(T) - V_1(H)/(S_1(H) - S_1(T)) = -1/2$ and $X_0 = (5/4)(.5\cdot 1.50 + .5\cdot -1.50) = 0$. So the bank's trader should shortsell 1/2 stock and we don't need any initial wealth to employ this strategy. We take the proceeds from this sale, $.5\cdot 4$, and invest in the money market. We can check our wealth at time 1 from H or T.

$$Y_1(H) = -1/2\cdot 8 + \frac{5}{4}(1/2\cdot 4) + 3 = 1.50$$

$$Y_1(T) = -1/2\cdot 2 + \frac{5}{4}(1/2\cdot 4) + 0 = 1.50$$

where we have accounted for the stock we must buy (as a result of the shortsell), the money from the shortsell we invested, and the value of the call. Regardless of the coin toss result, the portfolio value is 1.50.

1.7 Our goal here is to establish a portfolio that includes holding the lookback option, some stock and money market so that the value of the investment $Y_n = (5/4)^n 1.376, n = 1, 2, 3$. This is the risk-free return rate of the money market. This means that we must achieve $Y_{n+1}(H) = Y_{n+1}(T), n = 1, 2$, so

$$\begin{aligned} Y_{n+1}(H) &= \Delta_n S_{n+1}(H) + (1+r)(X_n - \Delta_n S_n) + V_{n+1}(H) \\ &= \Delta_n S_{n+1}(T) + (1+r)(X_n - \Delta_n S_n) + V_{n+1}(T) = Y_{n+1}(T). \end{aligned}$$

We perform a little algebra to find that we must choose Δ_n such that

$$\Delta_n = \frac{V_{n+1}(T) - V_{n+1}(H)}{S_{n+1}(H) - S_{n+1}(T)}.$$

This is just the δ-hedging strategy. Conversely, since the δ-hedging guarantees the fully invested portfolio to be risk-free (i.e. its payoff is independent of the market condition), it follows from the no-arbitrage argument that the portfolio must produce the risk-free return rate 1/4. Hence it must be the case that $Y_n = (5/4)^n 1.376, n = 1, 2, 3$.

For those who would like to see the actual numbers in action, we offer a check for the market scenario HHH. Note that $X_0 = 0$ since we are requiring $Y_0 = 1.376 = V_0$. Carrying out the actual computation, we have

$$\begin{aligned}
\Delta_0 &= \frac{V_1(T) - V_1(H)}{S_1(H) - S_1(T)} = \frac{1.20 - 2.24}{8 - 2} = -.1733 \\
\Delta_1(H) &= \frac{V_2(HT) - V_2(HH)}{S_2(HH) - S_2(HT)} = \frac{2.40 - 3.20}{16 - 4} = -.0666 \\
\Delta_1(T) &= \frac{V_2(TT) - V_2(TH)}{S_2(TH) - S_2(TT)} = \frac{2.20 - .80}{4 - 1} = .4666 \\
\Delta_2(HH) &= \frac{V_3(HHT) - V_3(HHH)}{S_3(HHH) - S_3(HHT)} = \frac{8 - 0}{32 - 8} = .3333 \\
\Delta_2(HT) &= \frac{V_3(HTT) - V_3(HTH)}{S_3(HTH) - S_3(HTT)} = \frac{6 - 0}{8 - 2} = 1 \\
\Delta_2(TH) &= \frac{V_3(THT) - V_3(THH)}{S_3(THH) - S_3(THT)} = \frac{2 - 0}{16 - 4} = .1666 \\
\Delta_2(TT) &= \frac{V_3(TTT) - V_3(TTH)}{S_3(TTH) - S_3(TTT)} = \frac{3.50 - 2}{2 - .50} = 1
\end{aligned}$$

and hence

$$\begin{aligned}
Y_1(H) &= X_1(H) + V_1(H) \\
&= -.1733 \cdot 8 + 1.25(0 - .1733 \cdot 4) + 2.24 = 1.7201 \\
Y_2(HH) &= X_2(HH) + V_2(HH) \\
&= -.0666 \cdot 16 + 1.25((1.7201 - 2.24) - (-.0666) \cdot 8) + 3.20 = 2.1504 \\
Y_3(HHH) &= X_3(HHH) + V_3(HHH) \\
&= .3333 \cdot 32 + 1.25((2.1504 - 3.20) - (.3333) \cdot 16) + 3.20 = 2.6876
\end{aligned}$$

so for the case HHH, we have achieved the desired growth (with some slight difference due to roundoff). One can show this for any sequence of coin tosses proceeding in a similar manner as above.

1.8 Consider an Asian call option.

(a) Suppose $\omega_1 \ldots \omega_n$ is a sequence of tosses that results in $S_n = s$ and $Y_n = y$. Then we should have $v_n(s, y) = V_n(\omega_1 \ldots \omega_n)$. From Theorem 1.2.2, we have

$$V_n(\omega_1 \ldots \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \ldots \omega_n T)].$$

For our specific model, this give

$$\frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \ldots \omega_n T)] = \frac{2}{5}[v_{n+1}(2s, y+2s) + v_{n+1}(s/2, y+s/2)].$$

So set $v_n(s,y) = \frac{2}{5}[v_{n+1}(2s, y+2s) + v_{n+1}(s/2, y+s/2)]$.

(b) We simply use our algorithm.

$$
\begin{aligned}
v_0(4,4) &= \frac{2}{5}(v_1(8,12) + v_1(2,6)) \\
v_1(8,12) &= \frac{2}{5}(v_2(16,28) + v_2(4,16)) \\
v_1(2,6) &= \frac{2}{5}(v_2(4,10) + v_2(1,7)) \\
v_2(16,28) &= \frac{2}{5}(v_3(32,60) + v_3(8,36)) \\
v_2(4,16) &= \frac{2}{5}(v_3(8,24) + v_3(2,18)) \\
v_2(4,10) &= \frac{2}{5}(v_3(8,18) + v_3(2,12)) \\
v_2(1,7) &= \frac{2}{5}(v_3(2,9) + v_3(1/2,15/2)) \\
v_3(32,60) &= 15 - 4 = 11 \\
v_3(8,36) &= 9 - 4 = 5 \\
v_3(8,24) &= 6 - 4 = 2 \\
v_3(2,18) &= 9/2 - 4 = 1/2 \\
v_3(8,18) &= 1/2 \\
v_3(2,12) &= v_3(2,9) = v_3(1/2,15/2) = 0
\end{aligned}
$$

Since we now know what information we need, we proceed forward.

$$
\begin{aligned}
v_2(1,7) &= \frac{2}{5}(v_3(2,9) + v_3(1/2,15/2)) = 0 \\
v_2(4,10) &= \frac{2}{5}(1/2 + 0) = 1/5 \\
v_2(4,16) &= \frac{2}{5}(2 + 1/2) = 1 \\
v_2(16,28) &= \frac{2}{5}(11 + 5) = 32/5 \\
v_1(2,6) &= \frac{2}{5}(1/5 + 0) = 2/25 \\
v_1(8,12) &= \frac{2}{5}(32/5 + 1) = 74/25 \\
v_0(4,4) &= \frac{2}{5}(74/25 + 2/25) = 152/125
\end{aligned}
$$

Thus, $v_0(4,4) = 152/125$.

(c) To compute $\delta_n(s,y)$, we apply formula (1.2.17) for some $\omega_1 \dots \omega_n$ such that $S_n = s$ and $Y_n = y$. That is,

$$
\begin{aligned}
\delta_n(s,y) &= \Delta_n(\omega_1 \dots \omega_n) = \frac{V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)}{S_{n+1}(\omega_1 \dots \omega_n H) - S_{n+1}(\omega_1 \dots \omega_n T)} \\
&= \frac{v_{n+1}(2s, y+2s) - v_{n+1}(s/2, y+s/2)}{2s - s/2}.
\end{aligned}
$$

We thus have our formula.