Math 133: Homework 4

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2.2 Consider the stock price S_3 in Figure 2.3.1.

(a) Find the distribution of S_3 under the risk-neutral probabilities of $\tilde{p} = 1/2, \tilde{q} = 1/2$.

Solution. A direct computation shows

$$\begin{aligned}
\tilde{\mathbb{P}}(S_3 = 1/2) = 1/8 \\
\tilde{\mathbb{P}}(S_3 = 2) = 3/8 \\
\tilde{\mathbb{P}}(S_3 = 8) = 3/8 \\
\tilde{\mathbb{P}}(S_3 = 32) = 1/8.
\end{aligned}$$

Most of this follows from Example 2.2.3. ☐

(b) Compute $\tilde{\mathbb{E}}S_1, \tilde{\mathbb{E}}S_2$, and $\tilde{\mathbb{E}}S_3$. What is the average rate of growth of the stock price under $\tilde{\mathbb{P}}$?

Solution. Again, a direct computation:

$$\begin{aligned}
\tilde{\mathbb{E}}S_1 &= \frac{1}{2}(8) + \frac{1}{2}(2) = 5 \\
\tilde{\mathbb{E}}S_2 &= \frac{1}{4}(16) + \frac{1}{2}(4) + \frac{1}{4}(1) = \frac{25}{4} \\
\tilde{\mathbb{E}}S_3 &= \frac{1}{8}(32) + \frac{3}{8}(8) + \frac{3}{8}(2) + \frac{1}{8}(.50) = \frac{125}{16}.
\end{aligned}$$

The formula for expectation is found in definition 2.2.4. ☐

(c) Answer the first two parts under the actual probabilities $p = 2/3, q = 1/3$.

Solution. A direct computation shows

$$\begin{aligned}
\mathbb{P}(S_3 = 1/2) &= \left(\frac{1}{3}\right)^3 = \frac{1}{27} \\
\mathbb{P}(S_3 = 2) &= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{6}{27} \\
\mathbb{P}(S_3 = 8) &= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{12}{27} \\
\mathbb{P}(S_3 = 32) &= \left(\frac{2}{3}\right)^3 = \frac{8}{27}.
\end{aligned}$$

Again, a direct computation:

$$\begin{aligned}
\mathbb{E}S_1 &= \frac{2}{3}(8) + \frac{1}{3}(2) = \frac{18}{3} = 6 \\
\mathbb{E}S_2 &= \frac{4}{9}(16) + \frac{2}{9}(4) + \frac{2}{9}(4) + \frac{1}{9}(1) = 9 \\
\mathbb{E}S_3 &= \frac{1}{27} \cdot \frac{1}{2} + \frac{6}{21} \cdot 2 + \frac{12}{27} \cdot 8 + \frac{8}{27} \cdot 32 = \frac{27}{2}.
\end{aligned}$$

The formula for expectation is found in definition 2.2.4. □

2.3 Show that a convex function of a martingale is a submartingale. In other words, let $M_0, \ldots, M_N$ be a martingale and let φ be a convex function. Show that $\varphi(M_0), \ldots, \varphi(M_N)$ is a submartingale.

Proof. This is immediate from conditional Jensen's inequality. Since $M_n = \tilde{\mathbb{E}}_n M_{n+1}$, Jensen's gives $\varphi(M_n) = \varphi(\tilde{\mathbb{E}}_n M_{n+1}) \leq \tilde{\mathbb{E}}_n \varphi(M_{n+1})$. □

2.4 Toss a coin repeatedly. Assume the probability of head on each toss is 1/2, as is the probability of tail. Let $X_j = 1$ if the jth toss results in a head and $X_j = -1$ if the jth toss results in a tail. Consider the stochastic process $M_0, M_1, M_2, \ldots$ defined by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, \;\; n \geq 1.$$

This is called a *symmetric random walk*; with each head, it steps up one and with each tail, it steps down one.

(a) Using the properties of Theorem 2.3.2, show that $M_0, M_1, M_2, \ldots$ is a martingale.

Proof. Since $M_{n+1} = M_n + X_{n+1}$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}_n[M_{n+1}] &= \tilde{\mathbb{E}}_n[M_n + X_{n+1}] = \tilde{\mathbb{E}}_n[M_n] + \tilde{\mathbb{E}}_n[X_{n+1}] \\
&= M_n + \mathbb{E}X_{n+1} = M_n + \frac{1}{2}(1) + \frac{1}{2}(-1) = M_n.
\end{aligned}$$

Thus, the sequence is a martingale. □

(b) Let σ be a positive constant and, for $n \geq 0$, define

$$S_n = e^{\sigma M_n} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n.$$

Show that $S_0, S_1, S_2, \ldots$ is a martingale.

Proof. Note that the following equalities all follow from properties of conditional expectation.

$$\begin{aligned}
\tilde{\mathbb{E}}_n[S_{n+1}] &= \tilde{\mathbb{E}}_n\left[e^{\sigma M_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\right] = \tilde{\mathbb{E}}_n\left[e^{\sigma(M_n+X_{n+1})}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\right] \\
&= \tilde{\mathbb{E}}_n\left[e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n} e^{\sigma X_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\right] \\
&= \tilde{\mathbb{E}}_n\left[S_n e^{\sigma X_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\right] = S_n\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\tilde{\mathbb{E}}_n[e^{\sigma X_{n+1}}] \\
&= S_n\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\left(\frac{e^{\sigma}+e^{-\sigma}}{2}\right) = S_n
\end{aligned}$$

Thus, the sequence is a martingale. □

2.5 Let $M_0, M_1, M_2, \ldots$ be the symmetric random walk, and define $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j), \; n = 1, 2, \ldots.$$

(a) Show that

$$I_n = \frac{1}{2}M_n^2 - \frac{n}{2}.$$

Proof. We proceed by induction. Note that we immediately have $I_0 = \frac{1}{2}M_0^2 - \frac{0}{2}$ for our base case. So now we just assume the statement holds for some n ($I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$) and show that the statement must also hold for $n+1$. Observe that $I_{n+1} = I_n + M_n(M_{n+1} - M_n)$. Thus, for any sequence of n tosses and H for the $n+1$ toss, we have

$$\begin{aligned}
I_{n+1}(H) &= I_n + M_n(M_{n+1}(H) - M_n) = I_n + M_n(M_n + 1 - M_n) \\
&= \frac{1}{2}M_n^2 - \frac{n}{2} + M_n \\
&= \frac{1}{2}(M_n^2 + 2M_n + 1) - \frac{1}{2} - \frac{n}{2} \\
&= \frac{1}{2}(M_n + 1)^2 - \frac{n+1}{2} = \frac{1}{2}M_{n+1}(H)^2 - \frac{n+1}{2}
\end{aligned}$$

A very similar calculation will show that this holds for T also.

$$\begin{aligned}
I_{n+1}(T) &= I_n + M_n(M_{n+1}(T) - M_n) = I_n + M_n(M_n - 1 - M_n) \\
&= \frac{1}{2}M_n^2 - \frac{n}{2} - M_n \\
&= \frac{1}{2}(M_n^2 - 2M_n + 1) - \frac{1}{2} - \frac{n}{2} \\
&= \frac{1}{2}(M_n - 1)^2 - \frac{n+1}{2} = \frac{1}{2}M_{n+1}(T)^2 - \frac{n+1}{2}
\end{aligned}$$

Since the equations agree for both H and T on the $n+1$ toss, we can conclude

$$I_{n+1} = \frac{1}{2}M_{n+1}^2 - \frac{n+1}{2}.$$

So by mathematical induction, the proof is complete. □

(b) Let n be an arbitrary nonnegative integer, and let $f(i)$ be an arbitrary function of a variable i. In terms of n and f, define another function $g(i)$ satisfying

$$\mathbb{E}_n[f(I_{n+1})] = g(I_n).$$

The conclusion is that $\{I_n\}_{n\geq 0}$ is a Markov process.

Proof. Noting $I_{n+1} = I_n + M_n(M_{n+1} - M_n) = I_n + M_n X_{n+1}$ where X_{n+1} is the Bernoulli random variable, we have

$$\begin{aligned}\mathbb{E}_n[f(I_{n+1})] &= \mathbb{E}_n[f(I_n + M_n X_{n+1})] \\ &= \frac{1}{2}f(I_n + M_n) + \frac{1}{2}f(I_n - M_n)\end{aligned}$$

Now from (i) we know that $M_n = \pm\sqrt{2I_n + n}$ whose sign is random. However the expression (1) is symmetrical w.r.t. $\pm M_n$ and hence we can write

$$\mathbb{E}_n[f(I_{n+1})] = g_n(I_n)$$

where

$$g_n(x) = \frac{1}{2}f(x + \sqrt{2x+n}) + \frac{1}{2}f(x - \sqrt{2x+n}).$$

□

2.6 Suppose $M_0, M_1, \ldots, M_N$ is a martingale, and let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted process. Define the *discrete-time stochastic integral* (sometimes called a *martingale transform*) $I_0, I_1, \ldots, I_N$ by setting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j),\ n = 1, \ldots, N.$$

Show that $I_0, I_1, \ldots, I_N$ is a martingale.

Proof. This is just a long set of equations. We are mostly using facts about conditional expectation, with the given assumptions that $M_0, M_1, \ldots, M_N$ is a martingale and $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ is an adapted process

$$\begin{aligned}\tilde{\mathbb{E}}_n[I_{n+1}] &= \tilde{\mathbb{E}}_n\left[\sum_{j=0}^{n-1}\Delta_j(M_{j+1} - M_j) + \Delta_n(M_{n+1} - M_n)\right] \\ &= \tilde{\mathbb{E}}_n\left[\sum_{j=0}^{n-1}\Delta_j(M_{j+1} - M_j)\right] + \tilde{\mathbb{E}}_n\left[\Delta_n(M_{n+1} - M_n)\right] \\ &= \tilde{\mathbb{E}}_n[I_n] + \tilde{\mathbb{E}}_n[\Delta_n M_{n+1}] - \tilde{\mathbb{E}}_n[\Delta_n M_n] \\ &= I_n + \Delta_n \tilde{\mathbb{E}}_n[M_{n+1}] - \Delta_n M_n \\ &= I_n + \Delta_n M_n - \Delta_n M_n = I_n\end{aligned}$$

Thus, the sequence is a martingale. □

2.8 Consider an N-period binomial model.

(a) Let $M_0, M_1, \ldots, M_N$ and $M'_0, M'_1, \ldots, M'_N$ be martingales under the risk-neutral probability measure. Show that if $M_N = M'_N$ (for every possible outcome of the sequence of coin tosses), then for each n between 0 and N, we have $M_n = M'_n$ (for every possible outcome of the sequence of coin tosses).

Proof. Since both sequences are martingales, we know

$$M_{N-1} = \mathbb{E}_{N-1}M_N = \mathbb{E}_{N-1}M'_N = M'_{N-1}.$$

Continuing inductively (backwards), we see that this holds for all n. □

(b) Let V_N be the payoff at time N of some derivaive security. This is a random variable that can depend on all N coin tosses. Define recursively $V_{N-1}, V_{N-2}, \ldots, V_0$ by the algorithm (1.2.16) of Chapter 1. Show that

$$V_0, \frac{V_1}{1+r}, \ldots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\tilde{\mathbb{P}}$.

Proof. Note that the definition of any given V_n is

$$V_n(\omega_1 \ldots \omega_n) = \frac{1}{1+r}(\tilde{p}V_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \ldots \omega_n T)) = \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}].$$

We now proceed

$$\tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right] = \frac{1}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{1+r}\right] = \frac{V_n}{(1+r)^n}$$

Therefore, the sequence is a martingale. □

(c) Using the risk-neutral pricing formula (2.4.11) of this chapter, define

$$V'_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right], \; n = 0, 1, \ldots, N-1.$$

Show that

$$V'_0, \frac{V'_1}{1+r}, \ldots, \frac{V'_{N-1}}{(1+r)^{N-1}}, \frac{V'_N}{(1+r)^N}$$

is a martingale.

Proof. We proceed by direct calculation.

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{V'_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-(n+1)}}\right]\right] \\
&= \frac{1}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-n}}\right]\right] \\
&= \frac{1}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] = \frac{V'_n}{(1+r)^n}
\end{aligned}$$

Therefore, the sequence is a martingale. □

(d) Conclude that $V_n = V_n'$ for every n.

Proof. This is almost immediate. We know that the last terms of the two martingales are equal. Thus all the terms are equal by part (i), i.e.,

$$\frac{V_n}{(1+r)^n} = \frac{V_n'}{(1+r)^n}.$$

Thus $V_n = V_n'$ for every n. □