(36) $\int \frac{e^{4t}+2e^{2t}-e^{t}}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^{t}-1}{e^{2t}+1}e^{t}\,dt$

Let $y = e^t \Rightarrow dy = e^t\,dt$

$\int \frac{y^3+2y-1}{y^2+1}\,dy = \int y + \frac{y-1}{y^2+1}\,dy$

$= \frac{1}{2}y^2 + \int \frac{y}{y^2+1}\,dy - \int \frac{1}{y^2+1}\,dy$

$= \frac{1}{2}y^2 + \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C$

$= \boxed{\frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C}$

(38) $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$. Let $y=\cos\theta \Rightarrow dy = -\sin\theta\,d\theta.$

$-\int \frac{1}{y^2+y-2}\,dy = -\int \frac{1}{(y-1)(y+2)}\,dy = \int \frac{A}{y-1} + \frac{B}{y+2}\,dy$

$\Rightarrow -1 = Ay + 2A + By - B$

$\Rightarrow A+B=0,\ 2A-B=-1 \Rightarrow 3A=-1 \Rightarrow A = -\frac{1}{3},$

$B = \frac{1}{3}.$

$-\frac{1}{3}\int \frac{1}{y-1}dy + \frac{1}{3}\int \frac{1}{y+2}\,dy = -\frac{1}{3}\ln|y-1| + \frac{1}{3}\ln|y+2| + C$

$= \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$

$= \boxed{\frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C}$

(43) $(t^2+2t)\frac{dx}{dt} = 2x+2$

$\Rightarrow \int \frac{1}{t^2+2t}\,dt = \int \frac{1}{2x+2}\,dx$

$\Rightarrow \int \frac{A}{t} + \frac{B}{t+2}\,dt = \frac{1}{2}\ln|x+1|$

$At + 2A + Bt = 1 \Rightarrow A+B=0,\ 2A=1 \Rightarrow A=\frac{1}{2},\ B=-\frac{1}{2}.$

$\frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| = \frac{1}{2}\ln|x+1|$

$\Rightarrow \frac{1}{2}\ln\left|\frac{t}{t+2}\right| = \frac{1}{2}\ln|x+1|$

$\Rightarrow \ln\left|\frac{t}{t+2}\right| + C = \ln|x+1|$

$t=1$ and $x=1 \Rightarrow \ln\left|\frac{1}{3}\right| + C = \ln 2 \Rightarrow C = \ln 2 - \ln\frac{1}{3}$

$\Rightarrow C = \ln 6$

$\Rightarrow \ln 6\left|\frac{t}{t+2}\right| = \ln|x+1| \Rightarrow x+1 = \frac{6t}{t+2}$

$\Rightarrow \boxed{x = \frac{6t}{t+2} - 1}$

(44) $(t+1)\frac{dx}{dt} = x^2+1 \Rightarrow \int \frac{1}{t+1}\,dt = \int \frac{1}{x^2+1}\,dx$

$\Rightarrow \tan^{-1}(x) = \ln|t+1| + C$

$t=0$ and $x=\frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{\pi}{4}\right) = \ln 1 + C$

$\Rightarrow C = \tan^{-1}\left(\frac{\pi}{4}\right)$

$\Rightarrow \underline{\tan^{-1}(x) = \ln|t+1| + \tan^{-1}\left(\frac{\pi}{4}\right)}$

$\Rightarrow \boxed{x = \tan\left(\ln|t+1| + \tan^{-1}\left(\frac{\pi}{4}\right)\right).}$

(50) $\frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k\,dt$

(a) $a=b \Rightarrow \int \frac{dx}{(a-x)^2} = \int k\,dt \Rightarrow \frac{1}{a-x} = kt + C$

$t=0$ and $x=0 \Rightarrow \frac{1}{a} = C$

$\Rightarrow \frac{1}{a-x} = kt + \frac{1}{a} \Rightarrow \frac{1}{a-x} = \frac{kta+1}{a}$

$\Rightarrow a-x = \frac{a}{kta+1} \Rightarrow \boxed{x = a - \frac{a}{kta+1}}$

(b) $a \neq b \Rightarrow \int \frac{A}{a-x} + \frac{B}{b-x}\,dx = \int k\,dt$

$Ab - Ax + aB - Bx = 1 \Rightarrow -A-B=0,\ Ab+aB=1$

$-A = B \Rightarrow -Bb + aB = 1 \Rightarrow B = \frac{1}{a-b},\ A = \frac{1}{b-a}.$

$\frac{1}{b-a}\int \frac{1}{a-x}\,dx + \frac{1}{a-b}\int \frac{1}{b-x}\,dx = kt + C$

$\Rightarrow \frac{-1}{b-a}\ln|a-x| - \frac{1}{a-b}\ln|b-x| = kt + C$

$x=0$ & $t=0 \Rightarrow \frac{-1}{b-a}\ln|a| - \frac{1}{a-b}\ln|b| = C$

So $\frac{1}{a-b}\ln\frac{|a-x|}{|b-x|} = kt + \frac{1}{a-b}\ln\left|\frac{a}{b}\right|$

$\Rightarrow \ln\left|\frac{a-x}{b-x}\right| = kt(a-b) + \ln\left|\frac{a}{b}\right|$

$\Rightarrow \frac{a-x}{b-x} = e^{kt(a-b)} \cdot \left|\frac{a}{b}\right| \Rightarrow a-x = ae^{kt(a-b)} - x\left(\frac{a}{b}\right)e^{kt(a-b)}$

$\Rightarrow a - ae^{kt(a-b)} = x - x\left(\frac{a}{b}\right)e^{kt(a-b)} = x\left(1 - \frac{a}{b}e^{kt(a-b)}\right)$

$\Rightarrow x = \frac{a(1-e^{kt(a-b)})}{1 - a/b\, e^{kt(a-b)}} = \boxed{\frac{ab(1-e^{kt(a-b)})}{b - ae^{kt(a-b)}}}$

(52) $\int \frac{P(x)}{x^3(x-1)^2}\,dx$

$P(x) = ax^2 + bx + C.$ $P(0) = 1 \Rightarrow C = 1.$

$P'(0) = 2a(0) + b = 0 \Rightarrow b = 0,$ so

$P(x) = ax^2 + 1.$

So $\int \frac{ax^2+1}{x^3(x-1)^2}\,dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}\,dx.$

For this to be rational, $A = 0$ and $D = 0.$

So $\int \frac{ax^2+1}{x^3(x-1)^2}\,dx = \int \frac{B}{x^2} + \frac{C}{x^3} + \frac{E}{(x-1)^2}\,dx$

$\Rightarrow ax^2 + 1 = Bx(x^2 - 2x + 1) + C(x^2 - 2x + 1) + Ex^3$

$\Rightarrow ax^2 + 1 = Bx^3 - 2Bx^2 + Bx + Cx^2 - 2Cx + C + Ex^3$

x^3: $B + E = 0$ x: $B - 2C = 0$

x^2: $-2B + C = a$ 1: $\boxed{C = 1}$

$\Rightarrow B - 2 = 0 \Rightarrow \boxed{B = 2}$

$\Rightarrow -2(2) + 1 = a \Rightarrow \boxed{a = -3}$

and $\boxed{E = -2}$

So $\boxed{P(x) = -3x^2 + 1}$

§8.4

(2) $\int_0^\pi \sin^5\left(\frac{x}{2}\right) dx = \int_0^\pi \sin\left(\frac{x}{2}\right) dx - \int_0^\pi 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx + \int_0^\pi \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx$

$= -2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right) \Big|_0^\pi$

$= -\left(-2 + \frac{4}{3} - \frac{2}{5}\right) = \frac{30}{15} - \frac{20}{15} + \frac{6}{15} = \boxed{1\frac{1}{15}}$

(12) $\int_0^\pi \cos^2 2x \sin 2x\,dx$ Let $u = \cos 2x \Rightarrow du = -2\sin 2x\,dx.$

So $-\frac{1}{2}\int_0^\pi u^2\,du = -\frac{1}{6}u^3\,du = -\frac{1}{6}\cos^3 2x \Big|_0^\pi$

$= -\frac{1}{6} + \frac{1}{6} = \boxed{0}$