Week 3

$\boxed{\S 5.4}$

(36) a) $\frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right)$

$= \frac{d}{dx}\left[\, t^3 \Big|_1^{\sin x}\right] = \frac{d}{dx}\left(\sin^3 x - 1\right)$

$= \boxed{3\sin^2 x \cos x}$

b) $\frac{d}{dx}\int_1^{\sin x} 3t^2\,dt$. Let $u = \sin x$, $du = \cos x\,dx$.

$\frac{d}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3u^2\cdot\frac{du}{dx} = 3u^2\cdot\cos x$

$= \boxed{3\sin^2 x\cos x.}$

(48) $y = \int_{2^x}^{1} \sqrt[3]{t}\,dt = -\int_1^{2^x}\sqrt[3]{t}\,dt.$

Let $u = 2^x \Rightarrow \frac{du}{dx} = \ln 2\cdot 2^x.$

So $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -u^{1/3}\cdot\ln 2\cdot 2^x$

$= -(2^x)^{1/3}\cdot 2^x\cdot\ln 2$

$= \boxed{-\ln 2\,(2^x)^{4/3}.}$

(50) $y = \int_{-1}^{x^{1/\pi}} \sin^{-1} t\,dt$

Let $u = x^{1/\pi} \Rightarrow \frac{du}{dx} = \frac{1}{\pi}x^{\frac{1-\pi}{\pi}}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \boxed{\sin^{-1}(x^{1/\pi})\cdot\frac{1}{\pi}x^{\frac{1-\pi}{\pi}}}$

$\boxed{\S 5.5}$

(2) $\int x\sin(2x^2)\,dx$, $u = 2x^2 \Rightarrow du = 4x\,dx$

So $\int x\sin(2x^2)\,dx = \frac{1}{4}\int \sin u\,du$

$= \frac{1}{4}(-\cos u) = \boxed{-\frac{1}{4}\cos(2x^2) + C}$

(10) $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$, $u = -\frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$

So $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx = \int \cos^2(-u)\,du$

$= \int\cos^2(u)\,du = \frac{u}{2} + \frac{1}{4}\sin 2u = \boxed{\frac{-1}{2x} + \frac{1}{4}\sin\left(\frac{-2}{x}\right) + C}$

(12) $\int \frac{dx}{\sqrt{5x+8}}$

a) $u = 5x+8 \Rightarrow du = 5\,dx$

So $\frac{1}{5}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{5}(2u^{1/2}) = \boxed{\frac{2}{5}\sqrt{5x+8} + C}$

b) $u = \sqrt{5x+8} = (5x+8)^{1/2} \Rightarrow du = \frac{5}{2}(5x+8)^{-1/2}$

So $\frac{2}{5}\int du = \frac{2}{5}u = \boxed{\frac{2}{5}\sqrt{5x+8} + C}$

(23) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$, $du = -\sin x\,dx$

So we have $-\int \frac{1}{u}\,du = -\ln|u| + C$

$= -\ln|\cos x| + C = \ln|\cos x|^{-1} + C$

$= \boxed{\ln|\sec x| + C}$

(34) $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

Let $u = \frac{1}{\sin\sqrt{\theta}} = \csc\sqrt{\theta} \Rightarrow du = \frac{1}{2\sqrt{\theta}}(-\csc\sqrt{\theta}\cot\sqrt{\theta}\,d\theta)$

So $\int \frac{\cos\sqrt{\theta}}{\sin\sqrt{\theta}} \cdot \frac{1}{\sqrt{\theta}} \cdot \frac{1}{\sin\sqrt{\theta}}\,d\theta = \int \cot\sqrt{\theta} \cdot \frac{1}{\sqrt{\theta}} \cdot \csc\sqrt{\theta}$

$= -2\int du = -2u + C = \boxed{-2\csc\sqrt{\theta} + C}$

(40) $\int (\sin 2\theta)\,e^{\sin^2\theta}\,d\theta$

Let $u = \sin^2\theta \Rightarrow du = 2\sin\theta\cos\theta = \sin 2\theta\,d\theta$.

Then $\int e^u\,du = e^u + C = \boxed{e^{\sin^2\theta} + C}$

(44) $\int \frac{\ln\sqrt{t}}{t}\,dt$

Let $u = \ln\sqrt{t} \Rightarrow du = \frac{1}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2t}\,dt$.

So $2\int u\,du = 2 \cdot \frac{1}{2}u^2 + C = u^2 + C = \boxed{(\ln\sqrt{t})^2 + C}$

$\boxed{\S 5.5}$ (cont'd)

(56) $\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\cos(x-1)\,dx$

a) $u = x-1 \Rightarrow du = dx$

$\int \sqrt{1+\sin^2 u}\ \sin u \cos u\ du$

$v = \sin u \Rightarrow dv = \cos u\ du$

$\int \sqrt{1+v^2}\ v\ dv$

$w = 1+v^2 \Rightarrow dw = 2v\ dv$

$\boxed{\frac{1}{2}\int \sqrt{w}\ dw}$

b) $u = \sin(x-1) \Rightarrow du = \cos(x-1)\ dx$

$\int \sqrt{1+u^2}\cdot u\ du$

$v = 1+u^2 \Rightarrow dv = 2u\ du$

$\boxed{\frac{1}{2}\int \sqrt{v}\ dv}$

c) $u = 1+\sin^2(x-1) \Rightarrow du = 2\sin(x-1)\cos(x-1)\ dx$

$\boxed{\frac{1}{2}\int \sqrt{u}\ du}$

(62) $\frac{dr}{d\theta} = 3\cos^2\left(\frac{\pi}{4}-\theta\right)$, $r(0) = \frac{\pi}{8}$

$\int 3\cos^2\left(\frac{\pi}{4}-\theta\right)d\theta$. Let $u = \frac{\pi}{4}-\theta \Rightarrow du = -d\theta$.

So $-\int 3\cos^2 u\ du = -3\left(\frac{u}{2}+\frac{1}{4}\sin 2u\right)+C$

$= \frac{-3}{2}\left(\frac{\pi}{4}-\theta\right) - \frac{3}{4}\sin 2\left(\frac{\pi}{4}-\theta\right) + C$

$r(0) = \frac{-3\pi}{8} - \frac{3}{4}\sin 2\left(\frac{\pi}{4}\right) + C = \frac{-3\pi}{8} - \frac{3}{4}\sin\frac{\pi}{2} + C = \frac{\pi}{8}$

$\Rightarrow C = \frac{\pi}{8} + \frac{3}{4}(1) + \frac{3\pi}{8}$

$r(\theta) = -\frac{3\pi}{8} + \frac{3}{2}\theta - \frac{3}{4}\sin\left(\frac{\pi}{2}-2\theta\right) + \frac{\pi}{8} + \frac{3}{4} + \frac{3\pi}{8}$

$= \boxed{\frac{3}{2}\theta - \frac{3}{4}\sin\left(\frac{\pi}{2}-2\theta\right) + \frac{\pi}{8} + \frac{3}{4}}$

(68) Both are correct: Note $\frac{\tan^2 x}{2} + C = \frac{\sec^2 x - 1}{2} + C$

$= \frac{\sec^2 x}{2} + \left(C - \frac{1}{2}\right)$,

So they differ only by a constant.

$\boxed{\S 5.6}$

(4) a) $\int_0^{\pi} 3\cos^2 x \sin x \; dx$

$= \int_0^{\pi} 3(-\cos x)^2 \cdot \frac{d}{dx}(-\cos x)\, dx$

$= \int_{-\cos 0}^{-\cos \pi} 3u^2\, du = u^3 \Big|_{-1}^{1} = 1-(-1) = \boxed{2}$

b) $\int_{2\pi}^{3\pi} 3(-\cos x)^2 \cdot \frac{d}{dx}(-\cos x)\, dx$

$= \int_{-\cos 2\pi}^{-\cos 3\pi} 3u^2\, du = u^3 \Big|_{-1}^{1} = \boxed{2}$

(10) a) $\int_0^1 \frac{x^3}{\sqrt{x^4+9}} = \int_0^1 \frac{1}{4} \frac{1}{\sqrt{x^4+9}} \cdot \frac{d}{dx}(x^4+9)$

$= \int_{0^4+9}^{1^4+9} \frac{1}{4}\frac{1}{\sqrt{u}}\, du = \frac{1}{4}\cdot 2\sqrt{u}\Big|_9^{10} = \boxed{\frac{1}{2}\sqrt{10} - \frac{3}{2}}$

b) $\int_{-1}^{0} \frac{x^3}{\sqrt{x^4+9}}\, dx = \frac{1}{4}\int_{10}^{9} \frac{1}{\sqrt{u}}\, du = -\frac{1}{4}\int_9^{10} \frac{1}{\sqrt{u}}\, du = \boxed{\frac{3}{2} - \frac{\sqrt{10}}{2}}$

(48) $y = (1-\cos x)\sin x$ from 0 to π

$\int_0^{\pi} \sin x - \sin x\cos x \; dx$

$= \int_0^{\pi} \sin x - \int_0^{\pi} \sin x \cos x \, dx$

$= -\cos x \Big|_0^{\pi} - \left(\frac{1}{2}\sin^2 x \Big|_0^{\pi}\right)$

$= 1-(-1) - \frac{1}{2}(0) = \boxed{2}$

(54) $\int_0^1 y^2 - y^3\, dy = \frac{1}{3}y^3 - \frac{1}{4}y^4\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$

(60) $$\int_{-2}^{0} 2x^3-x^2-5x-(-x^2+3x) + \int_{0}^{2} -x^2+3x-(2x^3-x^2-5x)$$

$$= \int_{-2}^{0} 2x^3-8x\,dx + \int_0^2 8x-2x^3\,dx$$

$$= \tfrac{1}{2}x^4-4x^2\Big|_{-2}^{0} + 4x^2-\tfrac{1}{2}x^4\Big|_0^2$$

$$= 0-\left(\tfrac{16}{2}-16\right) + 16-\tfrac{16}{2}-0$$

$$= -\left(-\tfrac{16}{2}\right)+\tfrac{16}{2}$$

$$= 2\left(\tfrac{16}{2}\right) = \boxed{16}$$

(106) $$\int_0^1 (x^{1/2}+1)-\left(\tfrac{x}{2}\right)^2 dx + \int_1^2 3-x-\left(\tfrac{x}{2}\right)^2 dx$$

$$= \tfrac{2}{3}x^{3/2}+x-\tfrac{x^3}{12}\Big|_0^1 + 3x-\tfrac{1}{2}x^2-\tfrac{x^3}{12}\Big|_1^2$$

$$= \tfrac{2}{3}+1-\tfrac{1}{12}-0 + \left(6-2-\tfrac{8}{12}\right)-\left(3-\tfrac{1}{2}-\tfrac{1}{12}\right)$$

$$= \tfrac{19}{12} + \tfrac{3}{2}-\tfrac{8}{12}+\tfrac{1}{12}$$

$$= 1+3/2 = \boxed{5/2}$$

$$\cancel{\int_0^2 3-y-(y-1)^2\,dy - \int_0^1 3-y-2\sqrt{y}\,dy}$$

$$\cancel{\int_0^2 3-y-(y^2-2y+1)\,dy - \int_0^1 3-y-2y^{1/2}\,dy}$$

$$\cancel{3y-\tfrac{y^2}{2}-\tfrac{y^3}{3}+y^2-y\Big|_0^2 - \left[3y-\tfrac{y^2}{2}-\tfrac{4}{3}y^{3/2}\Big|_0^1\right]}$$

$$\cancel{\left(6-2-\tfrac{8}{3}+4-2-0\right)-\left(3-\tfrac{1}{2}-\tfrac{4}{3}-0\right)}$$

$$\cancel{\left(6-\tfrac{8}{3}\right)-\left(\tfrac{5}{2}-\tfrac{4}{3}\right) = \tfrac{10}{3}-\tfrac{7}{6} = \tfrac{13}{6}}$$

-OR-

$$\int_1^2 3-y-(y-1)^2\,dy + \int_0^1 2\sqrt{y}-0\,dy$$

$$= 3y-\tfrac{1}{2}y^2-\tfrac{1}{3}y^3+y^2-y\Big|_1^2 + \tfrac{4}{3}y^{3/2}\Big|_0^1$$

$$= 6-2-\tfrac{8}{3}+4-2-\left(3-\tfrac{1}{2}-\tfrac{1}{3}+1-1\right)+\tfrac{4}{3}$$

$$= 6-\tfrac{8}{3}-\left(\tfrac{5}{2}-\tfrac{1}{3}\right)+\tfrac{4}{3} = 6-\tfrac{5}{2}-\tfrac{8}{3}+\tfrac{5}{3}$$

$$= \tfrac{7}{2}-1 = \boxed{5/2}$$

(110) Sometimes True (True only if $f(x) \geq g(x)\ \forall x$).