HW #4

§6.1 (6) Area $= (\text{side})^2$

Pyth. Thm $\Rightarrow 2\cdot(\text{side})^2 = (\text{diag})^2$, so $(\text{side})^2 = \frac{(\text{diag})^2}{2}$.

Area $= \frac{(\text{diag})^2}{2} = \frac{[\sqrt{1-x^2} - (-\sqrt{1-x^2})]^2}{2} = \frac{4(1-x^2)}{2}$

$= 2(1-x^2)$

Volume $= \int_{-1}^{1} 2(1-x^2)\,dx = 2\left[x - \frac{1}{3}x^3 \Big|_{-1}^{1}\right]$

$= 2\left[\left(\frac{2}{3}\right) - \left(-\frac{2}{3}\right)\right] = \boxed{\frac{8}{3}}$.

(10) Area $= \frac{1}{2}(\text{base})^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2(1-y^2)$;

Vol $= \int_{-1}^{1} 2(1-y^2)\,dy = \boxed{\frac{8}{3}}$.

(18) $V = \int_0^{\pi/2} \pi(\sin x \cos x)^2\,dx = \pi\int_0^{\pi/2} \frac{(\sin 2x)^2}{4}\,dx$

Let $u = 2x$, $du = 2\,dx$

So $V = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin^2 u}{4}\,du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{\sin 2u}{4}\Big|_0^{\pi/2}\right]$

$= \frac{2\pi x}{16} - \frac{\pi \sin 4x}{32}\Big|_0^{\pi/2} = \left(\frac{\pi^2}{16} - 0\right) - 0 = \boxed{\frac{\pi^2}{16}}$

(28) $V = \pi\int_0^{2\pi} y^2 \frac{dx}{d\theta}\,d\theta = \pi\int_0^{2\pi}(1-\cos\theta)^2(1-\cos\theta)\,d\theta$

$= \pi\int_0^{2\pi}(1-\cos\theta)^3\,d\theta$

$= \pi\int_0^{2\pi} 1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta\,d\theta$

Piece by piece: $\pi\int_0^{2\pi} d\theta = 2\pi^2$

$\pi\int_0^{2\pi} -3\cos\theta = -3\pi\sin\theta\Big|_0^{2\pi} = 0$

$\pi\int_0^{2\pi} 3\cos^2\theta\,d\theta = 3\pi\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{2\pi}\right)$

$= 3\pi^2$

$\pi\int -\cos^3\theta\,d\theta = \pi\int -\cos\theta(1-\sin^2\theta)\,d\theta$

$= 0$,

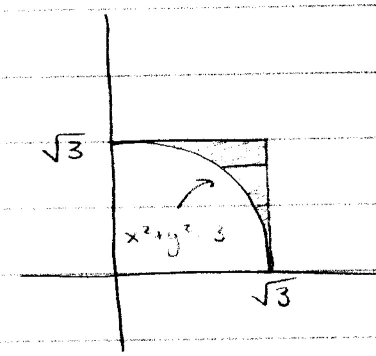
so $V = 2\pi^2 + 3\pi^2 = \boxed{5\pi^2}$.

(38) $\int_0^{\pi/4} \pi[1^2 - (\tan y)^2]\, dy = \pi \int_0^{\pi/4} 1 - (\sec^2 y - 1)\, dy$

$= \pi \int_0^{\pi/4} 2 - \sec^2 y\, dy = \pi\left[2y - \tan y \Big|_0^{\pi/4}\right]$

$= \pi\left(\frac{\pi}{2} - 1\right) = \boxed{\frac{\pi^2}{2} - \pi}$.

(48) $x^2 + y^2 = 3 \Rightarrow x = \sqrt{3 - y^2}$



$V = \pi \int_0^{\sqrt{3}} [(\sqrt{3})^2 - (\sqrt{3-y^2})^2]\, dy$

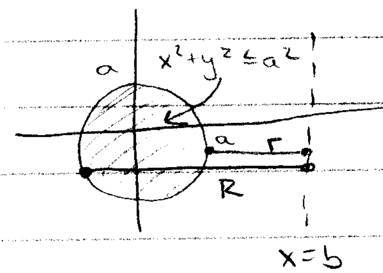
$= \pi \int_0^{\sqrt{3}} 3 - (3 - y^2)\, dy$

$= \pi \int_0^{\sqrt{3}} y^2\, dy = \frac{\pi}{3} y^3 \Big|_0^{\sqrt{3}}$

$= \boxed{\pi\sqrt{3}}$

(55) $R(y) = b + \sqrt{a^2 - y^2}$

$r(y) = b - \sqrt{a^2 - y^2}$



$V = \pi \int_{-a}^{a} (b + \sqrt{a^2 - y^2})^2 - (b - \sqrt{a^2 - y^2})^2\, dy$

$= \pi \int_{-a}^{a} b^2 + 2b\sqrt{a^2 - y^2} + a^2 - y^2\, dy$

$- \pi \int_{-a}^{a} b^2 - 2b\sqrt{a^2 - y^2} + a^2 - y^2\, dy$

$= \pi \int_{-a}^{a} b^2\, dy + \pi 2b \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy + \pi a^2 \int_{-a}^{a} dy - \pi \int_{-a}^{a} y^2\, dy$

$- \left[\pi b^2 \int_{-a}^{a} dy - 2b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy + \pi a^2 \int_{-a}^{a} dy - \pi \int_{-a}^{a} y^2\, dy\right]$

$= 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = 4b\pi\left(\frac{\pi a^2}{2}\right)$

$= \boxed{2a^2 b \pi^2}$.

(60) Area (circle) $= \pi\left[\sqrt{R^2 - h^2}\right]^2 = \pi(R^2 - h^2)$

Area (washer) $= \pi[R^2 - h^2] = \pi(R^2 - h^2)$

So Volume of Hemisphere = Volume Cylinder − Volume Cone

$\text{Vol}_{\text{Hem}} = \pi R^2 \cdot R - \frac{1}{3}\pi R^2 \cdot R$

$= \pi R^3 - \frac{1}{3}\pi R^3 = \boxed{\frac{2}{3}\pi R^3}$

§6.1 (63) $R(x) = |c - \sin x|$, so

a) $V = \pi \int_0^\pi (c - \sin x)^2 dx = \pi \int_0^\pi c^2 - 2c\sin x + \sin^2 x \, dx$

$= \pi \left[c^2 x + 2c \cos x + \frac{x}{2} - \frac{\sin 2x}{4} \Big|_0^\pi \right]$

$= \pi \left[c^2\pi + 2c(-1) + \frac{\pi}{2} - 0 - (0 + 2c(1)) \right]$

$= \pi \left(c^2\pi - 2c + \frac{\pi}{2} - 2c \right)$

$= c^2\pi^2 - 4c\pi + \frac{\pi^2}{2}.$

$\frac{dV}{dc} = 2\pi^2 c - 4\pi = 0$ when $2\pi^2 c = 4\pi$

$\Rightarrow c = \frac{2}{\pi}$ is a critical pt.

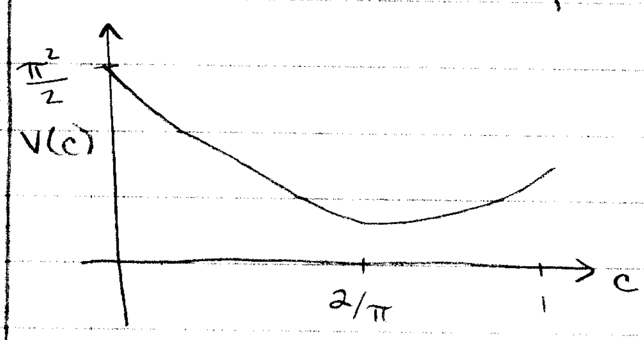
$V\left(\frac{2}{\pi}\right) = \frac{4}{\pi^2} \cdot \pi^2 - 8 + \frac{\pi^2}{2} = \frac{\pi^2}{2} - 4.$

$V(0) = \frac{\pi^2}{2}$ and $V(1) = \pi^2 - 4\pi + \frac{\pi^2}{2} = \frac{\pi^2}{2} - \pi(4 - \pi).$

$\boxed{\text{So abs. min is at } c = \frac{2}{\pi}, \text{ where } V = \frac{\pi^2}{2} - 4.}$

b) Max value at $c = 0$, $V = \frac{\pi^2}{2}$.

c)



As c moves away from 1, the volume increases w/o bound. This makes sense since our radius (dependent on c) also increases w/o bound.

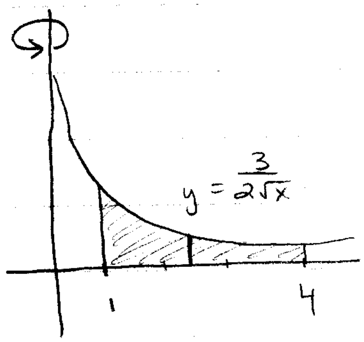
$\boxed{\S 6.2}$

(4) $V = \int_0^{\sqrt{3}} 2\pi (y) [3 - (3 - y^2)] \, dy$

$= 2\pi \int_0^{\sqrt{3}} y \cdot y^2 \, dy = 2\pi \int_0^{\sqrt{3}} y^3 \, dy$

$= \frac{\pi}{2} y^4 \Big|_0^{\sqrt{3}} = \boxed{\frac{9\pi}{2}}$

(12)



$V = 2\pi \int_1^4 (x)(\tfrac{3}{2} x^{-1/2}) \, dx$

$= 2\pi \int_1^4 \tfrac{3}{2} x^{1/2} \, dx$

$= 3\pi \left(\tfrac{2}{3} x^{3/2} \Big|_1^4 \right)$

$= 3\pi \left(\tfrac{2}{3} \cdot 8 - \tfrac{2}{3} \right) = \boxed{14\pi}$

(24) a) $V = 2\pi \int_0^2 (y) \left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2} \right) \right] dy$

$= 2\pi \int_0^2 y \left(y^2 - \tfrac{1}{4} y^4 \right) dy$

$= 2\pi \int_0^2 y^3 - \tfrac{1}{4} y^5 \, dy = 2\pi \left[\tfrac{1}{4} y^4 - \tfrac{1}{24} y^6 \Big|_0^2 \right]$

$= 2\pi \left[4 - \tfrac{8}{3} \right] = \boxed{\frac{8\pi}{3}}$

b) $V = 2\pi \int_0^2 (2 - y) \left(y^2 - \tfrac{1}{4} y^4 \right) dy$

$= 2\pi \int_0^2 2y^2 - \tfrac{1}{2} y^4 - y^3 + \tfrac{1}{4} y^5 \, dy$

$= 2\pi \left[\tfrac{2}{3} y^3 - \tfrac{1}{10} y^5 - \tfrac{1}{4} y^4 + \tfrac{1}{24} y^6 \Big|_0^2 \right]$

$= 2\pi \left[\tfrac{16}{3} - \tfrac{32}{10} - 4 + \tfrac{8}{3} \right] = 2\pi \left[8 - 4 - \tfrac{32}{10} \right]$

$= \boxed{\frac{8\pi}{5}}$

c) $V = 2\pi \int_0^2 (5 - y) \left(y^2 - \tfrac{1}{4} y^4 \right) dy = \boxed{8\pi}$

d) $V = 2\pi \int_0^2 \left(y + \tfrac{5}{8} \right) \left(y^2 - \tfrac{1}{4} y^4 \right) dy = \boxed{4\pi}$

§6.2

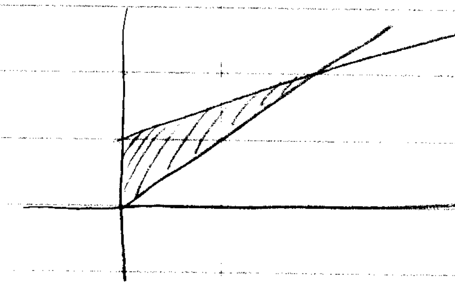
(26) a) $V = \pi \int_0^4 (\frac{1}{2}x+2)^2 - (x)^2 \, dx$

$= \pi \int_0^4 \frac{1}{4}x^2 + 2x + 4 - x^2 \, dx$

$= \pi \int_0^4 -\frac{3}{4}x^2 + 2x + 4 \, dx$

$= \pi \left[-\frac{1}{4}x^3 + x^2 + 4x \Big|_0^4 \right]$

$= \pi [-16 + 16 + 16] = \boxed{16\pi}$



b) $V = 2\pi \int_0^4 (x)(\frac{x}{2} + 2 - x) \, dx$

$= 2\pi \int_0^4 -\frac{x^2}{2} + 2x \, dx = 2\pi \left[-\frac{x^3}{6} + x^2 \Big|_0^4 \right]$

$= 2\pi \left(-\frac{32}{3} + \frac{48}{3} \right) = \boxed{\frac{32\pi}{3}}$

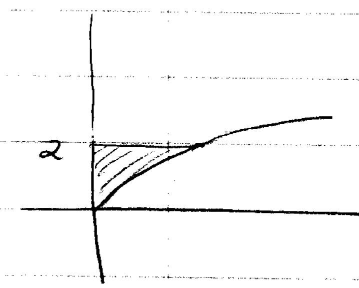
c) $V = 2\pi \int_0^4 (4-x)(-\frac{x}{2} + 2) \, dx = \boxed{\frac{64\pi}{3}}$

d) $V = \pi \int_0^4 \left[(8-x)^2 - (6 - \frac{x}{2})^2 \right] dx$

$= \boxed{48\pi}$.

(28) a) $V = 2\pi \int_0^2 (y)(y^2 - 0) \, dy$

$= 2\pi \int_0^2 y^3 \, dy = \boxed{8\pi}$



b) $V = 2\pi \int_0^2 (x)(2 - \sqrt{x}) \, dx$

$= 2\pi \int_0^2 2x - x^{3/2} \, dx$

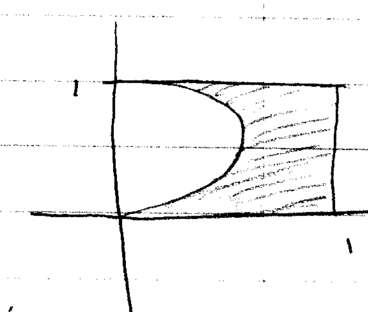
$= \boxed{\frac{32\pi}{5}}$

c) $V = 2\pi \int_0^2 (4-x)(2-\sqrt{x}) \, dx = \boxed{\frac{224\pi}{15}}$

d) $V = 2\pi \int_0^2 (2-y)(y^2 - 0) \, dy = \boxed{\frac{8\pi}{3}}$.

(30) a) $V = 2\pi \int_0^1 (y)(1 - y + y^3) \, dy$

$= \boxed{\frac{11\pi}{15}}$



b) $V = \pi \int_0^1 [1^2 - (y - y^3)^2] \, dy$

$= \boxed{\frac{97\pi}{105}}$

c) $V = \pi \int_0^1 [1 - (y - y^3)]^2 - 0^2 \, dy$

$= \boxed{\frac{121\pi}{210}}$

$\longrightarrow$

30 d) $V = 2\pi \int_0^1 (1-y)[1-y+y^3]\, dy = \boxed{\frac{23\pi}{30}}$

35 a) Disk Method: $V_1 - V_2$

$V_1 = \int_{-2}^{1} \pi \left(\sqrt{\frac{x+2}{3}}\right)^2 dx = \int_{-2}^{1} \frac{\pi}{3}(x+2)\, dx$

$V_2 = \int_0^1 \pi (\sqrt{x})^2\, dx = \int_0^1 \pi x\, dx$

$\rightarrow$ Two integrals needed.

b) Washer Method: $V_1 + V_2$

$V_1 = \pi \int_{-2}^{0} \left[\left(\sqrt{\frac{x+2}{3}}\right)^2 - 0^2\right] dx$

$V_2 = \pi \int_0^1 \left[\left(\sqrt{\frac{x+2}{3}}\right)^2 - (\sqrt{x})^2\right] dx$

$\rightarrow$ Two integrals needed

c) Shell Method:

$V = 2\pi \int_0^1 (y)(y^2 - 3y^2 + 2)\, dy$

$\rightarrow$ Only one integral needed.

All Methods Give $\boxed{\text{Volume} = \pi}$

36 a) Disk Method: $V_1 - V_2 - V_3$

$$\left.\begin{aligned} V_1 &= \pi \int_{-1}^{1} 1\, dy \\ V_2 &= \pi \int_0^1 \sqrt{y}\, dy \\ V_3 &= \pi \int_{-1}^{0} (-y)^{1/4}\, dy \end{aligned}\right\} \text{3 integrals}$$

b) Washer Method: $V_1 + V_2$

$$\left.\begin{aligned} V_1 &= \pi \int_0^1 1^2 - (\sqrt{y})^2\, dy \\ V_2 &= \pi \int_{-1}^{0} 1^2 - (-y^{1/4})^2\, dy \end{aligned}\right\} \text{2 integrals}$$

c) Shell Method:

$V = 2\pi \int_0^1 (x)(x^2 + x^4)\, dx \rightarrow 1$ integral

$\boxed{\text{Volume} = \frac{5\pi}{6}}$

$\boxed{\S 6.3}$

(8) $L = \int_0^{\pi} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$

$= \int_0^{\pi} \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2}\, dt$

$= \int_0^{\pi} \sqrt{e^{2t}\sin^2 t + e^{2t}\cos^2 t + e^{2t}\cos^2 t + e^{2t}\sin^2 t}\, dt$

$= \int_0^{\pi} \sqrt{2e^{2t}(\sin^2 t + \cos^2 t)}\, dt$

$= \int_0^{\pi} e^t \sqrt{2}\, dt = \sqrt{2}\, e^t \Big|_0^{\pi} = \boxed{\sqrt{2}(e^{\pi}-1)}$

(12) $L = \int_1^9 \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

$= \int_1^9 \sqrt{1 + (\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2})^2}\, dy$

$= \int_1^9 \sqrt{1 + \frac{1}{4}(y - 2 + \frac{1}{y})}\, dy = \int_1^9 \sqrt{\frac{1}{4}(y + 2 + \frac{1}{y})}\, dy$

$= \frac{1}{2}\int_1^9 \sqrt{(y^{1/2} + y^{-1/2})^2}\, dy = \frac{1}{2}\int_1^9 y^{1/2} + y^{-1/2}\, dy$

$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2} \Big|_1^9\right]$

$= \frac{1}{2}\left[18 + 6 - (\frac{2}{3} + 2)\right] = \frac{1}{2}\left[24 - (\frac{8}{3})\right] = \frac{1}{2}(\frac{64}{3}) = \boxed{\frac{32}{3}}$

(14) $L = \int_2^3 \sqrt{1 + (\frac{y^2}{2} - \frac{1}{2y^2})^2}\, dy$

$= \int_2^3 \sqrt{1 + \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4}y^{-4}}\, dy$

$= \int_2^3 \sqrt{\frac{1}{4}(2 + y^4 + y^{-4})}\, dy$

$= \frac{1}{2}\int_2^3 y^2 + y^{-2}\, dy = \frac{1}{2}\left[\frac{1}{3}y^3 - y^{-1} \Big|_2^3\right]$

$= \frac{1}{2}\left[9 - \frac{1}{3} - (\frac{8}{3} - \frac{1}{2})\right] = \frac{1}{2}\left[9 - 3 + \frac{1}{2}\right] = \boxed{\frac{13}{4}}$

(30) a) $L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy$; $(0,1)$

$(\frac{dx}{dy})^2 = \frac{1}{y^4} \Rightarrow \frac{dx}{dy} = \frac{1}{y^2} \Rightarrow x = -y^{-1} + C$

$\Rightarrow 0 = -1 + C \Rightarrow C = 1$, so $x = -\frac{1}{y} + 1$

$\Rightarrow \frac{1}{y} = 1 - x \Rightarrow \boxed{y = \frac{1}{1-x}}$.

b) Only one; we know deriv. of function and value at a point, so the curve is unique.

(32) $L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\, dx$; $(1,0)$

$\left(\frac{dy}{dx}\right)^2 = \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln x + C$

$\Rightarrow 0 = \ln 1 + C \Rightarrow C = 0$

$\boxed{y = \ln x}$

(34) $L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{(a - a\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{2a^2(1 - \cos\theta)}\, d\theta$

$= a\sqrt{2} \int_0^{2\pi} \sqrt{2} \sqrt{\frac{1-\cos\theta}{2}}\, d\theta$

$= a \cdot 2 \int_0^{2\pi} \sqrt{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= 2a \int_0^{2\pi} \left|\sin\frac{\theta}{2}\right| d\theta = 2a \int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$

$= 2a \left[-2\cos\frac{\theta}{2} \Big|_0^{2\pi} \right]$

$= 2a\left[-2(-1) - (-2)(1)\right]$

$= 2a\,[2+2] = \boxed{8a}$