$\boxed{\text{HW 5}}$: §6.4 (4, 8, 18, 22, 24, 28, 30, 40, 44)

(4) Rod 1 has length L; Rod 2 has length $2L$.
Center of mass of each rod is at the midpoint (by Example 1), so centers of mass are at coordinates $(\frac{L}{2}, 0)$ and $(0, L)$. So then

$$\bar{x} = \frac{\frac{L}{2}\cdot m + 0\cdot 2m}{m+2m} = \frac{Lm/2}{3m} = \frac{L}{6} \text{ and}$$

$$\bar{y} = \frac{0\cdot m + L\cdot 2m}{m+2m} = \frac{2mL}{3m} = \frac{2L}{3}$$

$\Rightarrow \left(\frac{L}{6}, \frac{2L}{3}\right)$ is the center of mass.

(8) $\delta(x) = 2 - \frac{x}{4}, \quad 0 \le x \le 4$

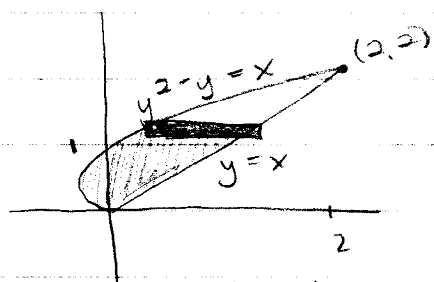
$$M_0 = \int_0^4 x\cdot\left(2 - \frac{x}{4}\right) dx = \int_0^4 2x - \frac{1}{4}x^2\, dx$$

$$= x^2 - \frac{1}{12}x^3 \Big|_0^4 = \left(16 - \frac{64}{12}\right) - 0 = \boxed{\frac{32}{3}}$$

$$M = \int_0^4 2 - \frac{x}{4}\, dx = 2x - \frac{1}{8}x^2 \Big|_0^4 = 8 - 2 = \boxed{6}$$

$$\bar{x} = \frac{M_0}{M} = \frac{32}{3\cdot 6} = \frac{32}{18} = \boxed{\frac{16}{9}}$$

(18)



Method of horiz. strips:

$$(\tilde{x}, \tilde{y}) = \left(\frac{(y^2-y)+y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$$

length $= y - (y^2 - y) = 2y - y^2$

width $= dy$

Area $=$ (length)$\cdot$(width) $= (2y - y^2)dy = dA$

mass $= \delta \cdot$ Area $= \delta(2y - y^2)\,dy = dm$

$$M_y = \int_0^2 \frac{1}{2}y^2 \cdot \delta(2y - y^2)\,dy = \int_0^2 \frac{\delta}{2}\cdot 2y^3 - y^4\,dy$$

$$= \frac{\delta}{2}\left(\frac{1}{2}y^4 - \frac{1}{5}y^5 \Big|_0^2\right) = \frac{\delta}{2}\left(8 - \frac{32}{5}\right) = \boxed{\frac{4\delta}{5}}$$

$$M_x = \int_0^2 \delta y\,(2y - y^2)\,dy = \int_0^2 \delta(2y^2 - y^3)\,dy = \delta\left(\frac{2}{3}y^3 - \frac{1}{4}y^4 \Big|_0^2\right)$$

$$= \delta\left(\frac{16}{3} - 4\right) = \frac{4\delta}{3}.$$

(cont'd) $\rightarrow$

$$M = \int_0^2 \delta(2y - y^2)\,dy = \delta\left(y^2 - \tfrac{1}{3}y^3 \Big|_0^2\right) = \delta\left(4 - \tfrac{8}{3}\right) = \frac{4\delta}{3}.$$

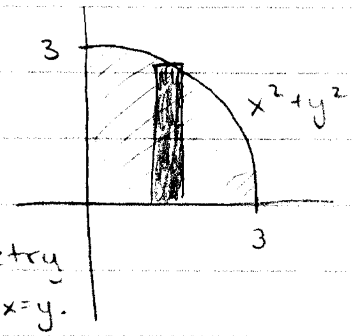
So $\bar{x} = \dfrac{M_y}{M} = \dfrac{4\delta}{5} \cdot \dfrac{3}{4\delta} = \dfrac{3}{5}$

$\bar{y} = \dfrac{M_x}{M} = \dfrac{4\delta}{3} \cdot \dfrac{3}{4\delta} = 1$

$$\boxed{(\bar{x}, \bar{y}) = \left(\tfrac{3}{5}, 1\right)}$$

(22) a)

Note: $\bar{x} = \bar{y}$ by symmetry about line $x = y$.



Vertical strips

$(\tilde{x}, \tilde{y}) = \left(x, \dfrac{\sqrt{9-x^2}}{2}\right)$

Length $= \sqrt{9-x^2}$

Width $= dx$

Area $= \sqrt{9-x^2}\,dx = dA$

Mass $= \delta\sqrt{9-x^2}\,dx = dm$

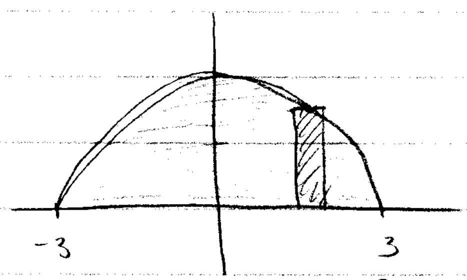
$$M_x = \int_0^3 \frac{\delta}{2}\left(\sqrt{9-x^2}\right)^2 dx = \int_0^3 \frac{\delta}{2}(9-x^2)\,dx$$
$$= \frac{\delta}{2}\left(9x - \tfrac{1}{3}x^3\Big|_0^3\right) = \frac{\delta}{2}(27-9) = 9\delta$$

$$M = \int_0^3 \delta\,dA = \delta(\text{Area of } \tfrac{1}{4} \text{ circle w/radius } 3)$$
$$= \delta \cdot \tfrac{1}{4}(\pi \cdot 3^2) = \tfrac{9}{4}\delta\pi.$$

So $\bar{y} = \dfrac{M_x}{M} = \dfrac{9\delta}{1} \cdot \dfrac{4}{9\delta\pi} = \dfrac{4}{\pi}$

$$\boxed{(\bar{x}, \bar{y}) = \left(\tfrac{4}{\pi}, \tfrac{4}{\pi}\right)}$$

b)



Vertical Strips have same parameters as in part (a).

So $$M_x = \int_{-3}^3 \frac{\delta}{2}(9-x^2)\,dx = \frac{\delta}{2}\left(9x - \tfrac{1}{3}x^3\Big|_{-3}^3\right)$$
$$= \frac{\delta}{2}\left[(27-9) - (-27+9)\right] = 18\delta$$

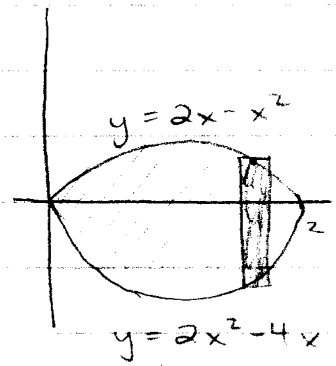
$$M = \int_{-3}^3 \delta\,dA = \delta(\tfrac{1}{2} \text{ Area of circle}) = \tfrac{9}{2}\delta\pi.$$

$\bar{y} = \dfrac{M_x}{M} = \dfrac{18\delta}{1} \cdot \dfrac{2}{9\delta\pi} = \dfrac{4}{\pi}.$

$\bar{x} = 0$ by symmetry of region about line $x = 0$.

So $\boxed{(\bar{x}, \bar{y}) = \left(0, \tfrac{4}{\pi}\right).}$

(24)



Vertical strips

$$(\tilde{x}, \tilde{y}) = \left(x, \frac{(2x - x^2) + (2x^2 - 4x)}{2}\right)$$
$$= \left(x, \frac{x^2 - 2x}{2}\right)$$

Region is symmetric about line $x = 1$, so $\bar{x} = 1$.

Length $= (2x - x^2) - (2x^2 - 4x) = 6x - 3x^2$

Width $= dx$

Area $= dA = 6x - 3x^2\, dx$

Mass $= \delta\,(6x - 3x^2)\, dx = dm$

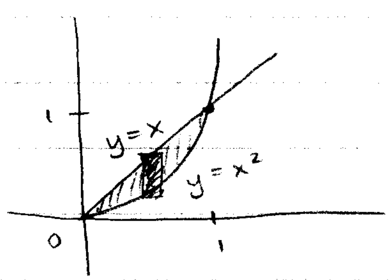
$$M_x = \int_0^2 \frac{\delta}{2}(x^2 - 2x)(6x - 3x^2)\, dx = \int_0^2 \frac{\delta}{2}(-3x^4 + 12x^3 - 12x^2)\, dx$$
$$= -\frac{3\delta}{2}\int_0^2 (x^4 - 4x^3 + 4x^2)\, dx = \frac{-3\delta}{2}\left(\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3 \Big|_0^2\right)$$
$$= -\frac{3\delta}{2}\left(\frac{32}{5} - 16 + \frac{32}{3}\right) = \frac{-3\delta}{2}\left(\frac{16}{15}\right) = \boxed{-\frac{8\delta}{5}}$$

$$M = \int_0^2 \delta\,(6x - 3x^2)\, dx = \delta\left(3x^2 - x^3 \Big|_0^2\right) = \delta(4) = \boxed{4\delta}$$

So $\bar{y} = \frac{M_x}{M} = \frac{-8\delta}{5} \cdot \frac{1}{4\delta} = -\frac{2}{5}$;

$$\boxed{(\bar{x}, \bar{y}) = \left(1, -\frac{2}{5}\right)}$$

(28)



Vertical strips

$$(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2 + x}{2}\right)$$

Length $= x - x^2$

Width $= dx$

Area $= dA = (x - x^2)\, dx$

Mass $= dm = \delta(x) \cdot dA = 12x(x - x^2)\, dx = 12x^2 - 12x^3\, dx$

$$M_x = \int_0^1 \frac{1}{2}(x^2 + x) \cdot (12x^2 - 12x^3)\, dx = 6\int_0^1 x^4 - x^5 + x^3 - x^4\, dx$$
$$= 6\int_0^1 x^3 - x^5\, dx = 6\left(\frac{1}{4}x^4 - \frac{1}{6}x^6 \Big|_0^1\right) = 6\left(\frac{1}{4} - \frac{1}{6}\right) = \boxed{\frac{1}{2}}$$

$$M_y = \int_0^1 x \cdot (12x^2 - 12x^3)\, dx = 12\int_0^1 x^3 - x^4\, dx = 12\left(\frac{1}{4}x^4 - \frac{1}{5}x^5 \Big|_0^1\right)$$
$$= 12\left(\frac{1}{4} - \frac{1}{5}\right) = \boxed{\frac{3}{5}}$$

$$M = \int_0^1 12x^2 - 12x^3\, dx = 12\left(\frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1\right) = 12\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{1}$$

So $\bar{x} = \frac{M_y}{M} = \frac{3}{5}$, $\bar{y} = \frac{M_x}{M} = \frac{1}{2}$, $\boxed{(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{1}{2}\right)}$

(30) (a) Disk Method: $\int_1^4 \pi\left(\frac{2}{x}\right)^2 dx = \int_1^4 \frac{4\pi}{x^2} dx$

$= 4\pi\left(-\frac{1}{x}\Big|_1^4\right) = 4\pi\left(-\frac{1}{4}+1\right) = \boxed{3\pi}$

(b) Vertical strips:

$(\tilde{x}, \tilde{y}) = \left(x, \frac{\left(\frac{2}{x}\right)}{2}\right) = \left(x, \frac{1}{x}\right)$

Length $= \frac{2}{x}$; Width $= dx$; Area $= dA = \frac{2}{x}\, dx$

Mass $= dm = \frac{2}{x} \cdot \delta(x)\, dx = \frac{2}{x}\sqrt{x}\, dx$

$M_x = \int_1^4 \frac{1}{x} \cdot \frac{2}{x}\sqrt{x}\, dx = \int_1^4 \frac{2x^{1/2}}{x^2} dx = 2\int_1^4 x^{-3/2} dx$

$= -4x^{-1/2}\Big|_1^4 = -4\left(\frac{1}{2}-1\right) = \boxed{2}$

$M_y = \int_1^4 x \cdot \frac{2}{x}\sqrt{x}\, dx = 2\int_1^4 x^{1/2} dx = \frac{4}{3}x^{3/2}\Big|_1^4$

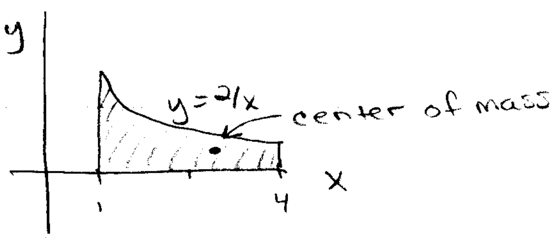
$= \frac{4}{3}(8-1) = \boxed{\frac{28}{3}}$

$M = \int_1^4 \frac{2}{x}\sqrt{x}\, dx = 2\int_1^4 x^{-1/2} dx = 4x^{1/2}\Big|_1^4$

$= 4(2-1) = \boxed{4}$

$\boxed{\text{So } (\bar{x}, \bar{y}) = \left(\frac{7}{3}, \frac{1}{2}\right)}$

(c)



(40) $y = x^3 \Rightarrow dy = 3x^2\, dx$

Length of a segment is $L = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + (3x^2 dx)^2}$

Mass $= \delta \cdot$ Length $= \delta\sqrt{(dx)^2 + 9x^4(dx)^2} = \delta \cdot dx\sqrt{1+9x^4}$

So $M_x = \int_0^1 \delta\, x^3\sqrt{1+9x^4}\, dx$

Let $u = 1+9x^4 \Rightarrow du = 36x^3\, dx$, so

$M_x = \delta\int_0^1 \frac{1}{36}\sqrt{u}\, du = \frac{\delta}{36} \cdot \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{\delta}{54}(1+9x^4)^{3/2}\Big|_0^1$

$= \boxed{\frac{\delta}{54}\left(10^{3/2}-1\right)}$

§6.4 (cont'd)

(44) $p > 0$; $y = \frac{x^2}{4p}$

WTS $\bar{y} = \frac{M_x}{M} = \frac{3}{5} a$ regardless of p value.

$(\tilde{x}, \tilde{y}) = (x, \frac{a + (x^2/4p)}{2})$

Length $= a - \frac{x^2}{4p}$, Width $= dx$, Area $= a - \frac{x^2}{4p} dx$

$y = a$ when $\frac{x^2}{4p} = a \Rightarrow x^2 = 4pa \Rightarrow x = \pm 2\sqrt{pa}$

By symmetry, $M_x = 2\int_0^{2\sqrt{pa}} \left(\frac{a}{2} + \frac{x^2}{8p}\right) \cdot \left(a - \frac{x^2}{4p}\right) \delta \, dx$

$= 2\delta \int_0^{2\sqrt{pa}} \frac{a^2}{2} - \frac{ax^2}{8p} + \frac{ax^2}{8p} - \frac{x^4}{32p^2} \, dx$

$= 2\delta \left[\frac{a^2}{2} x - \frac{1}{160p^2} x^5 \Big|_0^{2\sqrt{pa}} \right]$

$= 2\delta \left[a^2\sqrt{pa} - \frac{32}{160p^2} (pa)^2 \sqrt{pa} \right]$

$= 2\delta \left[a^2\sqrt{pa} - \frac{1}{5} a^2 \sqrt{pa} \right] = 2\delta \cdot \frac{4}{5} a^2 \sqrt{pa}$

$= \frac{8}{5} \delta a^2 \sqrt{pa}$

$M = 2\int_0^{2\sqrt{pa}} \delta \cdot \left(a - \frac{x^2}{4p}\right) dx = 2\delta \left[ax - \frac{x^3}{12p} \Big|_0^{2\sqrt{pa}} \right]$

$= 2\delta \left[2a\sqrt{pa} - \frac{8pa\sqrt{pa}}{12p} \right]$

$= 2\delta \left[2a\sqrt{pa} - \frac{2a\sqrt{pa}}{3} \right] = 2\delta \cdot \frac{2}{3} (2a\sqrt{pa})$

$= \frac{8a}{3} \delta \sqrt{pa}$

So $\bar{y} = \frac{8\delta a^2\sqrt{pa}}{5} \cdot \frac{3}{8a\delta\sqrt{pa}} = \boxed{\frac{3a}{5}}$.