HW 6

§ 6.5

(20) $x = \sqrt{2y-1}, \quad \frac{5}{8} \le y \le 1 \; ; \; y\text{-axis}$

$$S = \int_{5/8}^{1} 2\pi x \sqrt{1 + \left(\frac{1}{\sqrt{2y-1}}\right)^2}\, dy$$

$$= 2\pi \int_{5/8}^{1} \sqrt{2y-1} \cdot \sqrt{\frac{2y-1}{2y-1} + \frac{1}{2y-1}}\, dy$$

$$= 2\pi \int_{5/8}^{1} \sqrt{2y}\, dy$$

$$= 2\sqrt{2}\pi \left(\frac{2}{3} y^{3/2} \Big|_{5/8}^{1} \right) = 2\sqrt{2}\pi \left(\frac{2}{3}(1) - \frac{2}{3}\left(\frac{5\sqrt{5}}{16\sqrt{2}}\right)\right)$$

$$= \boxed{\frac{4}{3}\sqrt{2}\pi \left(1 - \frac{5\sqrt{5}}{16\sqrt{2}}\right)}$$

(24) $y = \cos x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \; ; \; x\text{-axis}$

$$S = \int_{-\pi/2}^{\pi/2} 2\pi y \sqrt{1 + \sin^2 x}\, dx$$

$$= \int_{-\pi/2}^{\pi/2} 2\pi (\cos x) \sqrt{1 + \sin^2 x}\, dx$$

$$\boxed{S = 2\pi \int_{-\pi/2}^{\pi/2} (\cos x) \sqrt{1 + \sin^2 x}\, dx}$$

(28) $$S = 2\pi \int_{a}^{a+h} y \cdot \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx$$

$$= 2\pi \int_{a}^{a+h} \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= 2\pi \int_{a}^{a+h} \sqrt{r^2 - x^2 + x^2}\, dx = 2\pi \int_{a}^{a+h} r\, dx$$

$$\boxed{S = 2\pi r h \quad \text{(independent of } a\text{)}}$$

(30) a) $x^2 + y^2 = 45^2 \Rightarrow x = \sqrt{45^2 - y^2} \Rightarrow \frac{dx}{dy} = \frac{-y}{\sqrt{45^2 - y^2}}$

$$S = 2\pi \int_{-22.5}^{45} \sqrt{45^2 - y^2} \cdot \sqrt{1 + \frac{y^2}{45^2 - y^2}}\, dy$$

$$= 2\pi \int_{-22.5}^{45} 45\, dy$$

$$= 2\pi(45)(45 + 22.5) = \boxed{6075\pi \text{ ft}^2}$$

b) $6075(3.1415) = 19084.6 \approx \boxed{19{,}085 \text{ ft}^2}$

(32) $x^{2/3} + y^{2/3} = 1 \Rightarrow y = (1-x^{2/3})^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\cdot\left(-\frac{2}{3}\right)x^{-1/3}\cdot\sqrt{1-x^{2/3}}$

$$S = 2\cdot 2\pi \int_0^1 (1-x^{2/3})^{3/2}\cdot\sqrt{1+\frac{1-x^{2/3}}{x^{2/3}}}\ dx$$

$$= 4\pi\int_0^1 (1-x^{2/3})^{3/2}\sqrt{\frac{1}{x^{2/3}}}$$

$$= 4\pi\int_0^1 (1-x^{2/3})^{3/2}\cdot x^{-1/3}\ dx$$

$$\left[\text{Let } u = 1-x^{2/3} \Rightarrow du = -\frac{2}{3}x^{-1/3}\ dx\right]$$

$$S = -6\pi\int_0^1 u^{3/2}\ du = -6\pi\left(\frac{2}{5}u^{5/2}\Big|_0^1\right)$$

$$= -6\pi\left[\frac{2}{5}(1-x^{2/3})^{5/2}\Big|_0^1\right] = -6\pi\left[\frac{2}{5}(0)-\frac{2}{5}(1)\right]$$

$$\boxed{S = \frac{12\pi}{5}}$$

(36) $x = a(t-\sin t)$, $y = a(1-\cos t)$, $0 \le t \le 2\pi$; x-axis

$\frac{dx}{dt} = a(1-\cos t)$, $\frac{dy}{dt} = a(\sin t)$

$$S = 2\pi\int_0^{2\pi} a(1-\cos t)\sqrt{a^2(1-2\cos t+\cos^2 t)+a^2\sin^2 t}\ dt$$

$$= 2\pi\int_0^{2\pi} a(1-\cos t)\sqrt{a^2-2a^2\cos t+a^2}\ dt$$

$$= 2\pi\int_0^{2\pi} a(1-\cos t)\cdot a\sqrt{2}\ \sqrt{1-\cos t}$$

$$\boxed{S = 2\pi a^2\sqrt{2}\int_0^{2\pi}(1-\cos t)^{3/2}\ dt}$$

(42) $S = \int_a^b 2\pi f(x)\ dx$ (Eqn. 8)

$$= \int_0^{\sqrt{3}} 2\pi\left(\frac{x}{\sqrt{3}}\right) dx$$

$$= \frac{2\pi}{\sqrt{3}}\left(\frac{1}{2}x^2\Big|_0^{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}\cdot\frac{1}{2}(3-0)$$

$$= \frac{3\pi}{\sqrt{3}} = \boxed{\sqrt{3}\ \pi}$$

(48) $S = 2\pi\rho L \Rightarrow 2\pi\left(a-\frac{2a}{\pi}\right)(\pi a)$

$$= 2\pi^2 a\left(\frac{\pi a-2a}{\pi}\right) = \boxed{2\pi a^2(\pi-2)}$$

(52) Line $\perp$ to $y = x - a$ and passing through $(0, \frac{2a}{\pi})$ is $y = -x + \frac{2a}{\pi}$.

Intersection of these two lines is where

$$x - a = -x + \frac{2a}{\pi} \Rightarrow 2x = \frac{2a}{\pi} + \frac{\pi a}{\pi}$$

$$\Rightarrow x = \frac{a(2+\pi)}{2\pi}$$

$$\Rightarrow y = -\left(\frac{2a + a\pi}{2\pi}\right) + \frac{2a}{\pi} = \frac{2a - a\pi}{2\pi}.$$

So distance from centroid to axis of rotation is

$$d = \sqrt{\left(\frac{a(2+\pi)}{2\pi}\right)^2 + \left(\frac{2a - a\pi}{2\pi} - \frac{2a}{\pi}\right)^2}$$

$$= \sqrt{\frac{a^2(4+4\pi+\pi^2)}{4\pi^2} + \left(\frac{-2a - a\pi}{2\pi}\right)^2}$$

$$= \sqrt{\frac{4a^2 + 4a^2\pi + a^2\pi^2 + 4a^2 + 4\pi a^2 + a^2\pi^2}{4\pi^2}}$$

$$= \sqrt{\frac{8a^2 + 8a^2\pi + 2a^2\pi^2}{4\pi^2}}$$

$$= \frac{a}{2\pi}\sqrt{8 + 8\pi + 2\pi^2} = \frac{a\sqrt{2}}{2\pi}\sqrt{\pi^2 + 4\pi + 4} = \frac{a\sqrt{2}}{2\pi}(\pi + 2).$$

So $S = 2\pi \rho L$

$$= 2\pi \cdot \frac{a\sqrt{2}}{2\pi}(\pi + 2)(\pi a)$$

$$\boxed{S = a^2\pi\sqrt{2}(\pi + 2).}$$

$\boxed{\S 6.6}$

(4) $F = kx$, so $90 \text{ N} = k(1 \text{ m}) \Rightarrow k = 90 \frac{\text{N}}{\text{m}}$.

Then $W = \int_0^5 kx \, dx = 90 \int_0^5 x \, dx = \frac{90}{2}(25) = \boxed{1125 \text{ J}}$

(11) Force against the piston is $F = pA$.

$V = Ax$, where x is ht. of the cylinder, then

$dV = A \, dx$.

$\Rightarrow W = \int F \, dx = \int pA \, dx = \int_{(p_1, V_1)}^{(p_2, V_2)} p \, dV$.

(12) $pV^{1.4} = c \Rightarrow p = cV^{-1.4}$.

So $V_1 = 243 \text{ in}^3$, $p_1 = 50 \text{ lb/in}^3 \Rightarrow c = 50(243)^{1.4} = 109{,}350 \text{ lb.}$

So $W = \int_{243}^{32} (109{,}350) V^{-1.4} \, dV$

$= \frac{109{,}350}{-.4} \cdot V^{-0.4} \Big|_{243}^{32} = \boxed{-37{,}968.75 \text{ in} \cdot \text{lb}}$

(13) Let r = constant rate of leakage.

Since bucket is leaking at constant rate and being lifted at a constant rate, then amt. of H_2O in bucket is proportional to $(20 - x)$. (the distance the bucket is being raised)

Leakage rate is $\frac{16 \text{ lbs}}{20 \text{ ft}} = .8$ lb/ft raised and wt. of H_2O in bucket is $F = .8(20 - x)$.

So $W = \int_0^{20} .8(20 - x) \, dx = 16x - .4x^2 \Big|_0^{20} = 320 - \frac{4}{10}(20^2)$

$= \boxed{160 \text{ ft} \cdot \text{lb}}$

(14) Same conditions as in #13 except leakage rate is $\frac{40 \text{ lbs}}{20 \text{ ft}} = 2 \text{ lb/ft}$ raised.

Wt. of H_2O in bucket is $F = 2(20-x)$, so

$$W = \int_0^{20} 2(20-x)\,dx = 2\left(20x - \tfrac{1}{2}x^2 \Big|_0^{20}\right)$$
$$= 2(400-200) = \boxed{400 \text{ ft}\cdot\text{lb}}$$

(18) Each "slab" of oil is to be pumped to a height of 14 ft. So the work to pump a slab is $(14-y)(\pi)\left(\frac{y}{2}\right)^2$. Since the tank is half full and $V_{original} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5^2)(10) = \frac{250\pi}{3} \text{ ft}^3$, then $V_{half} = \frac{250\pi}{6} \text{ ft}^3$. With half the volume the cone is filled to height y

$$\Rightarrow \frac{250\pi}{6} = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 \cdot y = \frac{\pi}{12}y^3 \Rightarrow y = \sqrt[3]{500} \text{ ft}.$$

So $W = \int_0^{\sqrt[3]{500}} \frac{57\pi}{4}(14y^2 - y^3)\,dy \approx \boxed{60{,}042 \text{ ft}\cdot\text{lb}}$

(20) Both methods need to pump H_2O 15 ft to base of tank. Pumping to the rim requires an additional 6 ft (so an avg. of 3 extra ft. to pump to the bottom of the tank).

Force of H_2O = Wt. of $H_2O \approx 62.4 \text{ lb/ft}^3$

So pumping through the valve uses

$(\sqrt{3} \text{ ft})(4\pi)\, 6 \text{ ft}^3 (62.4 \text{ lb/ft}^3) \approx 14{,}115 \text{ ft}\cdot\text{lb}$ less work and thus is faster.

(22) $y = x^2 \Rightarrow x = \sqrt{y}$

$\Delta V = \pi\,(\text{radius})^2\,(\text{thickness}) = \pi\,(\sqrt{y})^2\,\Delta y = \pi y\,\Delta y$

$F(y) = \text{weight} = 10{,}000\,\frac{N}{m^3} \cdot \Delta V = \left(10{,}000\,\frac{N}{m^3}\right)\pi\,(y\ m^2)\,\Delta y\ m$

$= 10{,}000\,\pi\, y\,\Delta y\ N.$

Ht of tank is $4^2 = 16$ m.

Distance to top of tank: $(16-y)$ m, so

$W = \int_0^{16} 10{,}000\,\pi y\,(16-y)\,dy$

$\approx \boxed{21{,}446{,}605.9\ J}$

§ 6.7

(2) Plate's right-hand edge given by
$y = x - 3. \Rightarrow x = y + 3.$
Total width is then $2x = 2(y+3) = 2y+6$.
Depth of a strip is $2-y$, so force of H_2O is given by

$$F = \int_{-3}^{0} \omega \cdot (\text{strip depth}) \cdot L(y)\, dy$$
$$= \int_{-3}^{0} (62.4)(2-y)(2y+6)\, dy$$
$$= 124.8 \int_{-3}^{0} (2-y)(y+3)\, dy$$
$$= 124.8 \int_{-3}^{0} 6 - y^2 - y\, dy = \boxed{1684.8 \text{ lb}}$$

(14) a) $V = (1000 \frac{ft^3}{h})(9\ h) = 9000\ ft^3$
$V = (\text{area of base})(\text{ht}) = (1500\ ft^2)\, h$
So $9000\ ft^3 = (1500 ft^2) h \Rightarrow h = 6$ ft.
So strip depth is $6-y$,
Total width is $2x = 2y = L(y)$,
So $F = 62.4 \int_0^1 (6-y)(2y)\, dy = \boxed{332.8 \text{ lb}}$

b) Want to find maximum h so that
$520 \text{ lb} = 62.4 \int_0^1 (h-y)(2y)\, dy$
$\Rightarrow 520 \text{ lb} = 124.8 \int_0^1 hy - y^2\, dy$
$\Rightarrow \frac{520}{124.8} = \frac{h}{2}y^2 - \frac{1}{3}y^3 \Big|_0^1 = \frac{h}{2} - \frac{1}{3}$
$\Rightarrow \boxed{h = 9 \text{ ft}}$

(15) Pressure at level y is $p = w \cdot h = w \cdot y$, so the avg. pressure

$$\bar{p} = \frac{1}{b}\int_0^b wy\, dy = \frac{1}{b}\left(\frac{w}{2}y^2 \Big|_0^b\right) = \frac{1}{b}\cdot\frac{w}{2}\cdot b^2 = \frac{bw}{2}.$$

(Note that this is the pressure at level $y = \frac{b}{2}$, which is the middle of the plate).

(16) $F = \int_a^b w\,(\text{depth})(\text{length})\, dy = \int_0^b w(y)(a)\, dy = aw\left(\frac{1}{2}y^2 \Big|_0^b\right) = \frac{awb^2}{2} = \left(\frac{bw}{2}\right)(ab) = \bar{p}\cdot \text{Area}.$

(17) When water reaches the top of the tank, the force on the moving side is

$$F = 62.4\int_{-2}^{0} \left(2\sqrt{4-y^2}\right)(-y)\, dy = 332.8 \text{ ft}\cdot\text{lb}.$$

The Force compressing the spring is $F = 100\cdot x$, so when the tank is full we have

$332.8 = 100x \Rightarrow x = 3.328$ ft.

Since $x < 5$, the moveable end will not reach the drain hole and the tank will overflow.

(22) $F = 62.4\int_0^{24} (24-y)\left(\frac{26}{24}\right)(100)\, dy = \boxed{1{,}946{,}880 \text{ lb}}$

(where $\frac{26}{24}\cdot 100\,\Delta y$ is the area of a strip of height Δy and parallel to the base of the dam).