HW #7

§7.1

(4) $\displaystyle\int \frac{8r\,dr}{4r^2-5}$

Let $u = 4r^2 - 5$. Then $du = 8r\,dr \Rightarrow \int \frac{1}{u}\,du \Rightarrow \ln|u| + C$

$= \boxed{\ln|4r^2-5| + C.}$

(10) $\displaystyle\int_{-\ln 2}^{0} e^{-x}\,dx = -e^{-x}\Big|_{-\ln 2}^{0} = -e^{0} - (-e^{\ln 2}) = -1 + 2 = \boxed{1}$

(16) $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\,dr$

Let $u = \sqrt{r}$. Then $du = \frac{1}{2\sqrt{r}}\,dr \Rightarrow \int 2e^{-u}\,du = -2e^{-u} + C$

$= \boxed{-2e^{-\sqrt{r}} + C}$

(34) $\displaystyle\int_1^2 \frac{2^{\ln x}}{x}\,dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx \Rightarrow \displaystyle\int_1^2 2^u\,du = \frac{2^u}{\ln 2}\Big|_1^2$

$= \displaystyle\frac{2^{\ln x}}{\ln 2}\Big|_1^2 = \frac{1}{\ln 2}\left(2^{\ln 2} - 2^0\right) = \boxed{\frac{1}{\ln 2}\left(2^{\ln 2} - 1\right)}$

(36) $\displaystyle\int_1^e x^{(\ln 2)-1}\,dx = \frac{x^{(\ln 2)-1+1}}{(\ln 2)-1+1}\Big|_1^e = \frac{x^{\ln 2}}{\ln 2}\Big|_1^e = \boxed{\frac{1}{\ln 2}\left(2 - 1\right)}$

$= \boxed{\frac{1}{\ln 2}}$

(50) $\dfrac{d^2y}{dt^2} = 1 - e^{2t}$, $y(1) = -1$ and $y'(1) = 0$.

$$\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$$

$$y'(1) = 0 \Rightarrow \frac{dy}{dt}(1) = 1 - \frac{1}{2}e^2 + C = 0 \Rightarrow C = \frac{1}{2}e^2 - 1.$$

So $$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1$$

$$\Rightarrow y(t) = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C$$

and $$y(1) = -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C$$

$$\Rightarrow -1 = -\frac{1}{2} + \frac{1}{2}e^2 + C$$

$$\Rightarrow C = -\frac{1}{2} - \frac{1}{2}e^2.$$

$$\therefore \boxed{y(t) = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \frac{1}{2}(1 + e^2).}$$

(51) $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}$, $y(1) = 3$.

$$\frac{dy}{dx} = 1 + \frac{1}{x} \Rightarrow y(x) = x + \ln|x| + C.$$

$$y(1) = 3 \Rightarrow 3 = 1 + \ln(1) + C \Rightarrow 3 = 1 + 0 + C \Rightarrow C = 2.$$

$$\therefore \boxed{y = x + \ln|x| + 2.}$$

~~(86)~~ $\cancel{x = \left(\frac{y}{4}\right)^2 - 2\ln\left(\frac{y}{4}\right), \quad 4 \le y \le 12}$

$$\cancel{L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_4^{12} \sqrt{1 + \left(\frac{y}{8} - \frac{2}{y}\right)^2}\, dy}$$

$$\cancel{= \int_4^{12} \sqrt{1 + \left(\frac{y^2}{64} - \frac{2y}{8y} + \frac{y}{16} - \frac{2y}{8y} + \frac{4}{y^2} - \frac{2}{2y} + \frac{y}{16} - \frac{2}{2y} + \frac{1}{4}\right)}\, dy}$$

$$\cancel{= \int_4^{12} \sqrt{1 + \frac{y^2}{64} - \frac{1}{4} + \frac{y}{8} - \frac{2}{y} + \frac{4}{y^2}}\, dy = \int_4^{12} \sqrt{\frac{64y^2 + \ldots y^4 - 16y^2 + 8y^2 - 32y + 256}{64y^2}}\, dy}$$

(56) $x = \frac{y^2}{16} - 2\ln\left(\frac{y}{4}\right) = \frac{y^2}{16} - 2\left[\ln y - \ln 4\right]$

$\frac{dx}{dy} = \frac{y}{8} - \frac{2}{y} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{64} - \frac{1}{2} + \frac{4}{y^2}$

$L = \int_a^b \sqrt{1 + (dx/dy)^2}\, dy = \int_4^{12} \sqrt{1 + \left(\frac{y}{8} - \frac{2}{y}\right)^2}\, dy$

$= \int_4^{12} \sqrt{1 + \frac{y^2}{64} - \frac{1}{2} + \frac{4}{y^2}}\, dy = \int_4^{12} \sqrt{\frac{32y^2 + 256 + y^4}{64y^2}}\, dy$

$= \int_4^{12} \sqrt{\frac{(y^2+16)^2}{64y^2}}\, dy = \int_4^{12} \frac{y^2+16}{8y}\, dy$

So $\int_4^{12} \frac{y}{8} + \frac{2}{y}\, dy = \frac{y^2}{16} + 2\ln|y| \Big|_4^{12} = \frac{144}{16} + 2\ln(12) - 1 - 2\ln(4)$

$= \frac{128}{16} + 2\left[\ln(12) - \ln(4)\right] = \boxed{\frac{128}{16} + 2\ln 3}$

(60) a) Using 2nd derivative test, we have $\frac{d^2}{dx^2}(e^x) = \frac{d}{dx}(e^x) = e^x$, which is positive for all x-values; hence, the graph of e^x is concave up everywhere.

b) Upper area estimate given by $\left(\text{Area}_{trap} = \frac{1}{2}b(h_1 + h_2)\right)$

$\frac{1}{2}\left(e^{\ln a} + e^{\ln b}\right) \cdot (\ln b - \ln a)$.

Lower area estimate given by

$\frac{1}{2}(AB + DC)(\ln b - \ln a)$, but $\frac{1}{2}(AB + DC) = f(M)$ since segment joining B to C is a line; thus we have

$e^{\frac{\ln a + \ln b}{2}}(\ln b - \ln a)$.

c) $\int_{\ln a}^{\ln b} e^x\, dx = b - a$, so (b) $\Rightarrow e^{\frac{\ln a + \ln b}{2}} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

$\Rightarrow e^{\frac{1}{2}\ln(ab)} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow (ab)^{1/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$.

$\boxed{\S 7.2}$

① a) $y = 0.99y_0$ when $t = 1000$, so

$y = y_0 e^{kt} \Rightarrow y = y_0(0.99) = y_0 e^{kt}$, so

$e^{kt} = 0.99 \Rightarrow kt = \ln(0.99) \Rightarrow \boxed{k = \frac{\ln(0.99)}{1000}}$.

b) $y(t) = .90y_0 = e^{\frac{\ln(0.99)}{1000}t} y_0$

$\Rightarrow .90 = e^{\frac{\ln(0.99)}{1000}t} \Rightarrow \ln .90 = \frac{\ln(0.99)}{1000} t$

$\Rightarrow \boxed{t = \frac{1000 \ln(.90)}{\ln(0.99)} \approx 10{,}536 \text{ years}}$

c) $y(20{,}000) = y_0 e^{\frac{\ln(0.99)}{1000} \cdot 20{,}000} = y_0 e^{20 \ln(0.99)} \approx \boxed{.82y_0 \Rightarrow 82\%}$

② a) $\frac{dp}{dh} = kp \Rightarrow \frac{dp}{p} = k\,dh$

$\Rightarrow \ln|p| = kh + C \Rightarrow p = e^{kh+c} = p_0 e^{kh} \Rightarrow 1013 e^{k(20)} = 90$

$\Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx \boxed{-0.121.}$

b) $p = 1013 e^{(-0.121)(50)} \approx \boxed{2.389 \text{ millibars}}$

c) $900 = 1013 e^{-0.121h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h \approx \boxed{0.977 \text{ km}}$

⑤ $L(x) = L_0 e^{-kx} \Rightarrow \frac{1}{2} L_0 = L_0 e^{-18k} \Rightarrow k \approx 0.0385.$

$\frac{1}{10} L_0 = L_0 e^{-0.0385x} \Rightarrow \boxed{x \approx 59.8 \text{ ft}}$

⑧ $10{,}000 = y_0 e^{3k}$ and $40{,}000 = y_0 e^{5k} \Rightarrow 4y_0 e^{3k} = y_0 e^{5k}$

$\Rightarrow 4 = e^{2k} \Rightarrow k = \ln 2.$

So $10{,}000 = y_0 e^{3\ln 2} \Rightarrow \boxed{y_0 = 1250}$

(9) a) $10,000\,e^{K(1)} = 7500 \Rightarrow e^{K} = .75 \Rightarrow K = \ln(.75)$.

Then $1,000 = 10,000\,e^{\ln(.75)t} \Rightarrow \boxed{t \approx 8 \text{ years}}$

b) $1 = 10,000\,e^{\ln(.75)t} \Rightarrow \boxed{t \approx 32.02 \text{ years}}$

(14) a) $A(t) = A_0 e^{t}$

b) $A(t) = 3A_0 = A_0 e^{t} \Rightarrow e^{t} = 3 \Rightarrow \boxed{t = \ln 3 \approx 1.098 \text{ years}}$

c) At the start of the year, you have $A_0 e^{t}$ dollars.
At the end of the year, you have $A_0 e^{t+1}$ dollars.
So the difference is $A_0 e^{t+1} - A_0 e^{t} = A_0 (e^{t+1} - e^{t})$

$= A_0 e^{t}(e-1) \approx 1.7(A_0 e^{t})$

$\boxed{\text{Approx. 1.7 times the amt. you start with.}}$

(15) $A(100) = 90,000 \Rightarrow 90,000 = 1000\,e^{r(100)} \Rightarrow 90 = e^{100r}$

$\Rightarrow 100r = \ln(90) \Rightarrow \boxed{r = \frac{\ln 90}{100} \approx .045}$

(16) $A(100) = 131,000 \Rightarrow 131,000 = 1000e^{100r} \Rightarrow 131 = e^{100r}$

$\Rightarrow \boxed{r = \frac{\ln 131}{100} \approx .04875}$

(20) a) $\frac{1}{2} y_0 = y_0 e^{K(2.645)} \Rightarrow 2.645K = -\ln 2 \Rightarrow \boxed{K = \frac{-\ln 2}{2.645} \approx -.262}$

b) $\frac{1}{|K|} \approx \boxed{3.816 \text{ years}}$

c) $.05 y_0 = y_0 e^{-.262t} \Rightarrow -.262t = \ln(.05) \Rightarrow \boxed{t = \frac{\ln(.05)}{-.262} \approx 11.431 \text{ yrs}}$

(24) Let x = how far above room temp silver will be in 15 mins,

y = " in 120 mins,

t_0 = time silver will be 10°C above room temp.

Then we can set up the following table:

Time in mins.	0	20 (Now)	35	140	t_0
Temp	$T_s + 70°$	$T_s + 60°$	$T_s + x$	$T_s + y$	$T_s + 10°$

Newton's Law $\Rightarrow T - T_s = (T_0 - T_s)e^{kt} \Rightarrow (60 + T_s) - T_s = [(70 + T_s) - T_s]e^{-20k}$

$\Rightarrow 60 = 70e^{-20k} \Rightarrow \boxed{k = (-\frac{1}{20})\ln(\frac{6}{7}) \approx .00771}$

a) $T - T_s = (T_0 - T_s)e^{-.00771t} \Rightarrow (T_s + x) - T_s = [(70 + T_s) - T_s]e^{-.00771(35)}$

$\Rightarrow \boxed{x = 70e^{-.26985} \approx 53.44°C}$

b) $(T_s + y) - T_s = [(70 + T_s) - T_s]e^{-.00771(140)}$

$\Rightarrow \boxed{y = 70e^{-1.0794} \approx 23.79°C}$

c) $(T_s + 10) - T_s = [(70 + T_s) - T_s]e^{-.00771(t_0)}$

$\Rightarrow 10 = 70e^{-.00771t_0} \Rightarrow \ln(\frac{1}{7}) = -.00771t_0$

$\Rightarrow \boxed{t_0 \approx 252.38 \text{ mins}}$

(26) a) Using $k \approx -.0001216$ (from Example 5), we have

$.17y_0 = y_0e^{-.0001216t} \Rightarrow \frac{\ln(.17)}{-.0001216} = t \approx 14,572$ yrs

$\Rightarrow \boxed{12,572 \text{ BC}}$

b) $t = \frac{\ln(.18)}{-.0001216} \approx 14,102 \Rightarrow \boxed{12,102 \text{ BC}}$

c) $t = \frac{\ln(.16)}{-.0001216} \approx 15,071 \Rightarrow \boxed{13,071 \text{ BC}}$