21 B - 3rd Midterm

① $\int_1^3 \frac{3^{\ln x}}{x}\,dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx.$

$$\int_1^3 3^u\,du = \frac{1}{\ln 3}\cdot 3^u \Big|_{\ln 1}^{\ln 3} = \frac{1}{\ln 3}\left(3^{\ln 3} - 3^0\right)$$

$$= \boxed{\frac{1}{\ln 3}\left(3^{\ln 3} - 1\right)}$$

$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx.$

$$\int 2e^{-u}\,du \Rightarrow -2e^{-u} + C = \boxed{-2e^{-\sqrt{x}} + C}$$

② $\frac{d^2y}{dt^2} = 1 - e^t$, $y(1) = 1$, $y'(1) = 0.$

$\frac{d^2y}{dt^2} = 1 - e^t \Rightarrow \frac{dy}{dt} = t - e^t + C$

Use $y'(1) = 0$: $y'(1) = 0 = 1 - e^1 + C \Rightarrow C = e - 1.$

So $\frac{dy}{dt} = t - e^t + e - 1 \Rightarrow y = \frac{1}{2}t^2 - e^t + t\cdot e - t + C$

Use $y(1) = 1$: $y(1) = 1 = \frac{1}{2}(1)^2 - e^{(1)} + 1\cdot e - 1 + C$

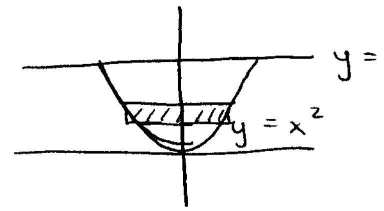
$$1 = \frac{1}{2} - e + e - 1 + C$$

$$1 = \frac{1}{2} - 1 + C$$

$$1 = -\frac{1}{2} + C \Rightarrow C = \frac{3}{2}.$$

So $\boxed{y(t) = \frac{1}{2}t^2 - e^t + te - t + \frac{3}{2}}$

③ $p = gh \Rightarrow F = \int_a^b g \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) \cdot L(y)\, dy$



$$\Rightarrow F = \int_0^1 g \cdot (1-y) \cdot 2\sqrt{y}\; dy$$

$$= 2g \int_0^1 y^{1/2} - y^{3/2}\; dy$$

$$= 2g \left[\tfrac{2}{3} y^{3/2} - \tfrac{2}{5} y^{5/2} \Big|_0^1 \right]$$

$$= 2g\left(\tfrac{2}{3} - \tfrac{2}{5}\right) = 2g\left(\tfrac{4}{15}\right) = \boxed{\tfrac{8}{15}\, g \text{ units}}$$

④ $\Delta V = \pi r^2 \Delta y = \pi(\sqrt{y})^2 \Delta y = \pi y \Delta y$

$F(y) = 10{,}000 \cdot \pi y \Delta y$ N

$\Delta W = (1-y)\, 10{,}000\, \pi y \Delta y$

$$10{,}000\pi \int_0^1 (1-y) y\, dy = 10{,}000\pi \left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 \Big|_0^1\right] = \boxed{\frac{10{,}000\pi}{6} = \frac{5{,}000\pi}{3}} \quad \text{N-m}$$

⑤ $S = \int_a^b 2\pi f(x) \sqrt{1 + (dy/dx)^2}\; dx$

$$= \int_{-\pi/2}^{\pi/2} 2\pi (\cos x) \sqrt{1 + (-\sin x)^2}\; dx$$

$$\boxed{= \int_{-\pi/2}^{\pi/2} 2\pi \cos x \sqrt{1+\sin^2 x}\; dx}$$

⑥ $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$

$$= \int_0^{2\pi} 2\pi (1-\cos t)\sqrt{(1-\cos t)^2 + (\sin t)^2}\; dt$$

$$\boxed{= \int_0^{2\pi} 2\pi (1-\cos t)\sqrt{2 - 2\cos t}\; dt}$$

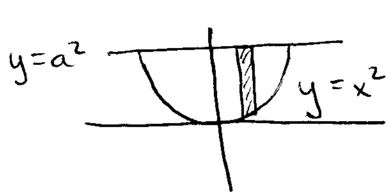
⑦ $\tilde{x} = x, \quad \tilde{y} = \frac{1}{2}(x^2 + a^2)$

Length $= a^2 - x^2$

Width $= dx$

$dA = a^2 - x^2\, dx$

$dm = \delta(a^2 - x^2)dx$



$$\bar{x} = \frac{M_y}{M} = \frac{\int \tilde{x}\, \delta(a^2-x^2)dx}{\int \delta(a^2-x^2)dx} = \frac{\int (x)\;(a^2-x^2)\,dx}{\int a^2-x^2\,dx} = \ldots$$

(by symmetry, we know $\bar{x} = 0$)

$$\bar{y} = \frac{M_x}{M} = \frac{\int \tilde{y}\,\delta(a^2-x^2)dx}{\int \delta(a^2-x^2)dx} = \frac{\frac{1}{2}\int (x^2+a^2)(a^2-x^2)dx}{\int a^2-x^2\,dx} = \frac{\frac{1}{2}\int a^4 - x^4\,dx}{\int a^2-x^2\,dx}$$

$$= \frac{\frac{1}{2}\left(a^4 x - \frac{1}{5}x^5 \Big|_{-a}^{a}\right)}{a^2 x - \frac{1}{3}x^3 \Big|_{-a}^{a}} = \frac{\frac{4}{5}a^5}{\frac{4}{3}a^3} = \frac{4}{5}a^2 \cdot \frac{3}{4} = \frac{3}{5}a^2.$$

$\bar{y} = \frac{3}{5}a^2 \qquad \boxed{(\bar{x}, \bar{y}) = \left(0, \tfrac{3}{5}a^2\right)}$