§ 1.6 $\boxed{30.a}$

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 3 & 3 & 6 & 1 \\ 5 & 1 & -8 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 3 & 6 & 1 \\ 4 & 2 & -1 & 5 \\ 5 & 1 & -8 & 8 \end{array}\right] \xrightarrow{R_1 \div 3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 4 & 2 & -1 & 5 \\ 5 & 1 & -8 & 8 \end{array}\right]$$

$$\xrightarrow[:R_2]{-4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 0 & -2 & -9 & 11/3 \\ 5 & 1 & -8 & 8 \end{array}\right] \xrightarrow[:R_3]{-5R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 0 & -2 & -9 & 11/3 \\ 0 & -4 & -18 & 19/3 \end{array}\right] \xrightarrow[:R_3]{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1/3 \\ 0 & -2 & -9 & 11/3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Notice $0 \neq -1$

$\Rightarrow$ NO SOLUTION

$\boxed{30.b}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 2 & 1 & -3 & 3 \\ 1 & 0 & 2 & -1 & -1 \end{array}\right] \xrightarrow[:R_3]{-R_1+R_3} \left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 2 & 1 & -3 & 3 \\ 0 & -1 & -1 & 2 & -1 \end{array}\right] \xrightarrow[:R_2]{2R_3+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[:R_2]{-1R_2} \left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[:R_2]{+R_3+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 3 & -3 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[:R_1]{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[:R_1]{3R_3+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[:R_3]{-R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$$\Rightarrow \begin{aligned} x_1 + x_4 &= 1 \\ x_2 - x_4 &= 2 \\ x_3 - x_4 &= -1 \end{aligned}$$

$$\boxed{\text{Let } x_4 = s,\ s \in \mathbb{R} \Rightarrow \begin{aligned} x_1 &= 1 - s \\ x_2 &= 2 + s \\ x_3 &= -1 + s \\ x_4 &= s \end{aligned}}$$

§ 1.6 $\boxed{34}$ $f: \mathbb{R}^3 \to \mathbb{R}^3$ st

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{aligned} x + 2y + 3z &= a \\ -3x - 2y - z &= b \\ -2x \quad\quad + 2z &= c \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ -3 & -2 & -1 & b \\ -2 & 0 & 2 & c \end{array}\right] \xrightarrow[2R_1+R_3 : R_3]{3R_1+R_2 : R_2} \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 8 & 3a+b \\ 0 & 4 & 8 & 2a+c \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3 : R_3} \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 8 & 3a+b \\ 0 & 0 & 0 & -a-b+c \end{bmatrix}$$

$\boxed{\text{This system is only consistent if } -a-b+c=0}$

§ 1.6 $\boxed{36}$ $A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -1 \\ 4 & -2 & 3 \end{bmatrix}$ $b_1 = \begin{bmatrix} 3 \\ -7 \\ 12 \end{bmatrix}$ $b_2 = \begin{bmatrix} -4 \\ 6 \\ -10 \end{bmatrix}$

1) $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ -3 & 2 & -1 & -7 \\ 4 & -2 & 3 & 12 \end{array}\right] \xrightarrow[\text{RREF (see steps below)}]{\text{find}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ for $Ax = b_1$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

2) $\left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ -3 & 2 & -1 & 6 \\ 4 & -2 & 3 & -10 \end{array}\right] \xrightarrow[\text{RREF (see steps below)}]{\text{find}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$ for $Ax = b_2$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

3) $\left[\begin{array}{ccc|c|c} 1 & -2 & 0 & 3 & -4 \\ -3 & 2 & -1 & -7 & 6 \\ 4 & -2 & 3 & 12 & -10 \end{array}\right] \xrightarrow[-4R_1+R_3 : R_3]{3R_1+R_2 : R_2} \left[\begin{array}{ccc|c|c} 1 & -2 & 0 & 3 & -4 \\ 0 & -4 & -1 & 2 & -6 \\ 0 & 6 & 3 & 0 & 6 \end{array}\right]$

$\xrightarrow[R_3 \div 6 : R_3]{\frac{1}{2}(R_3+R_2) : R_2} \left[\begin{array}{ccc|c|c} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3 : R_3} \left[\begin{array}{ccc|c|c} 1 & -2 & 0 & 3 & -4 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & 1 \end{array}\right]$

$\xrightarrow[\substack{2R_3+R_2 : R_2 \\ -2R_3 : R_3}]{2R_2+R_1 : R_1} \left[\begin{array}{ccc|c|c} 1 & 0 & 2 & 5 & -4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right] \xrightarrow{-2R_3+R_1 : R_1} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right]$ ★

SAME SOLN'S!

★ Note: results are the same in all 3 augmented matrices since steps for getting rref are same for all three.

§ 1.6 $\boxed{T5}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Show A row equiv to $I_2 \Leftrightarrow ad-bc \neq 0$

$\mapsto$ note: $c[a \;\; b] = [ca \;\; cb] = [ac \;\; bc]$

and $a[c \;\; d] = [ac \;\; ad]$ $\quad (a,b,c,d \in \mathbb{R})$

Assume A row equiv to I_2

If $ad-bc=0$ Thus, $ad = bc$

$\Rightarrow R_1$ + R_2 of A are multiples of each other

since $c \times R_1 = [ac \;\; bc] = [ac \;\; ad] = a \times R_2$

$\Rightarrow$ elementary row operations on A will produce a matrix with rows that are multiples of each other

$\Rightarrow$ elementary row ops cannot produce I_2 if $ad=bc$

$\Rightarrow$ A not row equiv to I_2 (contradiction!)

$\therefore ad-bc \neq 0$

$\leftarrow$ If $ad-bc \neq 0$, then (WLOG) not both $a, c = 0$

Assume $a \neq 0$

Consider $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$ and, applying row ops

$$\xrightarrow[-c \times R_1 + R_2 : R_2]{R_1 \div a} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \times \frac{a}{ad-bc}} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{1}{ad-bc} \end{array}\right]$$

$$\xrightarrow{R_2 \times \frac{-b}{a} + R_1 : R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$\Rightarrow A$ row equiv. to I_2, as required ∎

Note: $\forall \theta$, not both $\cos\theta = \sin\theta = 0$ WLOG assume $\cos\theta \neq 0$

§1.6 $\boxed{T7}$ $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \xrightarrow{R_1 \div \cos\theta} \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\xrightarrow[: R_2]{R_1 \times \sin\theta + R_2} \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} \end{bmatrix}$$

note: $\cos\theta + \dfrac{\sin^2\theta}{\cos\theta}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta}$

$$= \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ 0 & \frac{1}{\cos\theta} \end{bmatrix}$$

$$\xrightarrow{R_2 \times \cos\theta : R_2} \begin{bmatrix} 1 & \frac{\sin\theta}{\cos\theta} \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow[: R_1]{R_2 \times \frac{-\sin\theta}{\cos\theta} + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now assume $\cos\theta = 0$, thus $\sin\theta \neq 0$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$\xrightarrow{R_1 \div -\sin\theta} \begin{bmatrix} 1 & \frac{-\cos\theta}{\sin\theta} \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$\xrightarrow[: R_2]{R_1 \times (-\cos\theta) + R_2} \begin{bmatrix} 1 & \frac{-\cos\theta}{\sin\theta} \\ 0 & \sin\theta + \frac{\cos^2\theta}{\sin\theta} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\cos\theta/\sin\theta \\ 0 & 1/\sin\theta \end{bmatrix}$$

$$\xrightarrow[: R_2]{R_2 \times \sin\theta} \begin{bmatrix} 1 & \frac{-\cos\theta}{\sin\theta} \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow[: R_1]{R_2 \times \frac{\cos\theta}{\sin\theta} + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\boxed{\text{Thus } \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \xrightarrow{RREF} I_2}$$

§ 1.6 $\boxed{T8}$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Show $Ax=0$ has only trivial soln $\Leftrightarrow$ $ad-bc \neq 0$

Note that there are only 4 possibilities for RREF form ~~of~~ A

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

by Cor 1-1 $A \xrightarrow[RREF]{} B$

$\Rightarrow$ A, B have same solutions

$\mapsto$ If $Ax=0$ has only trivial sol'n

$\Rightarrow B = I_2$, since all other forms clearly have non-trivial solutions

$\Rightarrow ad-bc \neq 0$ by $\boxed{T5}$

$\leftarrow$ if $ad-bc \neq 0$,

$\boxed{T5} \Rightarrow$ A row equiv to $I_2 \Rightarrow Ax=0$ has only the trivial sol'n.

as req'd ■

§ 1.6 $\boxed{T10}$ For homogeneous system

$$\left.\begin{array}{l} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{array}\right\}$$

Find λ st system has non-trivial sol'n satisfying $(a-\lambda)(d-\lambda) - bc = 0$

Let $A = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

By $\boxed{T8}$, A has <u>only</u> <u>trivial</u> sol'n $\Leftrightarrow$

$(a-\lambda)(d-\lambda) - bc \neq 0$

$\Rightarrow$ for A to have non-trivial sol'n

$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$, as req'd ■

§ 1.6 $\boxed{T9}$ Let A be in RREF and assume $A \neq I_n$

Then there exists at least one row of A without a leading "1"

By definition of RREF it follows this row must be a zero row.

■

§1.6 $\boxed{T12}$ Show u, v soln's to $Ax=b$

$\Rightarrow u-v$ a soln to $Ax=0$

u, v soln's to $Ax=b$

$\Rightarrow Au=b$ and $Av=b$

Consider
$$\begin{aligned} A(u-v) &= Au - Av \\ &= b - b \\ &= 0 \end{aligned}$$

$\Rightarrow u-v$ a soln to $Ax=0$, as req'd ∎

§1.7 $\boxed{8}$ (a)

$$\underset{A}{\left[\begin{matrix}1&1&1\\1&2&3\\0&1&1\end{matrix}\right.}\left|\underset{I_3}{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}\right] \xrightarrow{-R_1+R_2:R_2} \left[\begin{matrix}1&1&1\\0&1&2\\0&1&1\end{matrix}\right|\left.\begin{matrix}1&0&0\\-1&1&0\\0&0&1\end{matrix}\right]$$

$$\xrightarrow{-R_2+R_3:R_3} \left[\begin{matrix}1&1&1\\0&1&2\\0&0&-1\end{matrix}\right|\left.\begin{matrix}1&0&0\\-1&1&0\\1&-1&1\end{matrix}\right] \xrightarrow[\substack{R_3+R_1:R_1\\ -1R_3:R_3}]{2R_3+R_2:R_2} \left[\begin{matrix}1&1&0\\0&1&0\\0&0&1\end{matrix}\right|\left.\begin{matrix}2&-1&1\\1&-1&2\\-1&1&-1\end{matrix}\right]$$

$$\xrightarrow{-R_2+R_1:R_1} \left[\underset{I_3}{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}\right|\left.\begin{matrix}1&0&-1\\1&-1&2\\-1&1&-1\end{matrix}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix}1&0&-1\\1&-1&2\\-1&1&-1\end{bmatrix}}$$

(b)

$$\underset{A}{\left[\begin{matrix}1&2&2\\1&3&1\\1&3&2\end{matrix}\right.}\left|\underset{I_3}{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}\right] \xrightarrow[-R_1+R_3:R_3]{-R_1+R_2:R_3} \left[\begin{matrix}1&2&2\\0&1&-1\\0&1&0\end{matrix}\right|\left.\begin{matrix}1&0&0\\-1&1&0\\-1&0&1\end{matrix}\right]$$

$$\xrightarrow[-R_2+R_3:R_3]{-2R_2+R_1:R_1} \left[\begin{matrix}1&0&4\\0&1&-1\\0&0&1\end{matrix}\right|\left.\begin{matrix}3&-2&0\\-1&1&0\\0&-1&1\end{matrix}\right] \xrightarrow[-4R_3+R_1:R_1]{R_3+R_2:R_2} \left[\underset{I_3}{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}\right|\left.\begin{matrix}3&2&-4\\-1&0&1\\0&-1&1\end{matrix}\right]$$

$A^{-1}=$

(c)

$$\left[\begin{matrix}1&2&3\\1&1&2\\0&1&1\end{matrix}\right|\left.\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] \xrightarrow{-R_1+R_2:R_2} \left[\begin{matrix}1&2&3\\0&-1&-1\\0&1&1\end{matrix}\right|\left.\begin{matrix}1&0&0\\-1&1&0\\0&0&1\end{matrix}\right]$$

$$\xrightarrow{R_2+R_3:R_3} \left[\begin{matrix}1&2&3\\0&-1&-1\\0&0&0\end{matrix}\right|\left.\begin{matrix}&&\\&&\\&&\end{matrix}\right]$$

A cannot be made into I_3 by row operations $\Rightarrow$ $\boxed{A \text{ singular}}$

§ 1.7 $\boxed{12}$ (a)
$$\left.\begin{aligned} x+y+2z&=0\\ 2x+y+z&=0\\ 3x-y+z&=0\end{aligned}\right\} \quad \left[\begin{array}{ccc|c}1&1&2&0\\2&1&1&0\\3&-1&1&0\end{array}\right] \xrightarrow[-3R_1+R_3:R_3]{-2R_1+R_2:R_2} \left[\begin{array}{ccc|c}1&1&2&0\\0&-1&-3&0\\0&-4&-5&0\end{array}\right]$$

$$\xrightarrow{4R_2+R_3:R_3} \left[\begin{array}{ccc|c}1&1&2&0\\0&-1&-3&0\\0&0&-17&0\end{array}\right] \xrightarrow[-R_2:R_2]{R_3\div -17:R_3} \left[\begin{array}{ccc|c}1&1&2&0\\0&1&3&0\\0&0&1&0\end{array}\right] \xrightarrow{-3R_3+R_2:R_2} \left[\begin{array}{ccc|c}1&1&2&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$$\xrightarrow[-2R_3+R_1:R_1]{-R_2+R_1:R_1} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$\boxed{\text{(a)} \xrightarrow[RREF]{} I_3 \Rightarrow \text{only trivial sol'n}}$

(b)
$$\left[\begin{array}{ccc|c}1&-1&1&0\\2&1&0&0\\2&-2&2&0\end{array}\right] \xrightarrow{-2R_1+R_3:R_3} \left[\begin{array}{ccc|c}1&-1&1&0\\2&1&0&0\\0&0&0&0\end{array}\right]$$

(R_3 is a mult of R_1)

$\boxed{\text{(b)} \not\xrightarrow[RREF]{} I_3 \Rightarrow \text{has non-trivial sol'n}}$

(c)
$$\left[\begin{array}{ccc|c}2&-1&5&0\\3&2&-3&0\\1&-1&4&0\end{array}\right] \xrightarrow[-3R_3+R_2:R_2]{-R_3+R_1:R_1} \left[\begin{array}{ccc|c}1&0&1&0\\0&5&-15&0\\1&-1&4&0\end{array}\right] \xrightarrow[R_2\div 5:R_2]{-R_1+R_3:R_3} \left[\begin{array}{ccc|c}1&0&1&0\\0&1&-3&0\\0&-1&3&0\end{array}\right]$$

$$\xrightarrow{R_2+R_3:R_3} \left[\begin{array}{ccc|c}1&0&1&0\\0&1&-3&0\\0&0&0&0\end{array}\right]$$

$\boxed{\text{(c)} \not\rightarrow I_3 \Rightarrow \text{has non-trivial sol'n}}$

(one row is a linear combination of the other 2)

§ 1.7 $\underline{16}$

$$\underbrace{\left[\begin{array}{ccc}1&1&0\\1&0&0\\1&2&a\end{array}\right.}_{A}\underbrace{\left|\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right]}_{I_3} \xrightarrow[-R_1+R_3:R_3]{-R_1+R_2:R_2} \left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&-1&0&-1&1&0\\0&1&a&-1&0&1\end{array}\right]$$

$$\xrightarrow[R_2+R_3:R_3]{R_2+R_1:R_1} \left[\begin{array}{ccc|ccc}1&0&0&0&1&0\\0&-1&0&-1&1&0\\0&0&a&-2&1&1\end{array}\right] \xrightarrow[\frac{1}{a}R_3:R_3]{-R_2:R_2} \underbrace{\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right.}_{I_3}\underbrace{\left|\begin{array}{ccc}0&1&0\\1&-1&0\\-\frac{2}{a}&\frac{1}{a}&\frac{1}{a}\end{array}\right]}_{A^{-1}}$$

Inverse exists $\forall\ a \in \mathbb{C},\ a \neq 0$

(or $a \in \mathbb{R}, a \neq 0$ if restricted to real matrices)

$$A^{-1} = \begin{bmatrix}0&1&0\\1&-1&0\\-2/a&1/a&1/a\end{bmatrix}$$

§ 1.7 $\boxed{18}$

(a) $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2 : R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$

$(A) \quad (I_2)$

$\xrightarrow[:R_1]{-3R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$

$(I_2) \quad (A^{-1})$

(b) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right]$

$(A^T) \quad (I_2)$

$\xrightarrow[:R_1]{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right]$

$(I_2) \quad ((A^T)^{-1})$

$$\boxed{A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = (A^{-1})^T}$$

§ 1.7 $\boxed{T1}$ A, B square matrices st $AB = 0$

If B non-singular, find A

Note that B non-sing $\Rightarrow \exists B^{-1}$

$\therefore AB = 0 \Rightarrow (AB)B^{-1} = 0(B^{-1})$

$\Rightarrow A(BB^{-1}) = 0$

$\Rightarrow A(I_n) = 0$

$\Rightarrow \boxed{A = 0}$, the $n \times n$ zero matrix

§ 1.7 $\boxed{T4}$ Show $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ non-singular $\Leftrightarrow ad-bc \neq 0$

By Th 1.12, A non-singular $\Leftrightarrow A$ row-equivalent to I_2, and

by § 1.6 $\boxed{T5}$ A row-equiv to $I_2 \Leftrightarrow ad-bc \neq 0$

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

(See § 1.6 $\boxed{T5}$ for calculation)

§ 1.7 $\boxed{T3}$ A is row equivalent to a matrix B in RREF which by Theorem 1.12 is not I_n. $\blacksquare$

Thus B has fewer than n nonzero rows and fewer than n unknowns corresponding to pivotal columns of B. Choose one of the free unknowns – unknowns not corresponding to pivotal columns of B. Assign any nonzero value to that unknown. This leads to a nontrivial solution to the homogenous system $A\underline{x} = 0$.

§ 1.7 T8 Show A non-singular & symmetric $\Rightarrow A^{-1}$ symmetric

A non-singular $\Rightarrow \exists A^{-1}$

Consider $(A^{-1})^T = (A^T)^{-1} = (A)^{-1}$ since A symmetric

$\Rightarrow A^{-1}$ symmetric, as required ■

§ 1.7 T9 A diagonal with non-zero $a_{11}, a_{22}, \dots, a_{nn}$. Show A^{-1} non-singular, diagonal matrix with diagonal entries $1/a_{11}, 1/a_{22}, \dots, 1/a_{nn}$

A diagonal with $a_{11}, a_{22}, \dots, a_{nn}$ non-zero

$\Rightarrow a_{ij} \neq 0$ if $i=j$, and $a_{ij} = 0$ otherwise.

Need $n \times n$ matrix B st $AB = I_n$

$\Rightarrow$ need $(ab)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = 1$ if $i=j$, and 0 otherwise

$\Rightarrow b_{ii} = \frac{1}{a_{ii}}$ and $b_{ij} = 0$ if $i \neq j$

$\Rightarrow A$ non-singular & $A^{-1} = B$ ■

§1.8 1,3 on next page.

§ 1.8 6 $A = \begin{bmatrix} -3 & 1 & -2 \\ -12 & 10 & -6 \\ 15 & 13 & 12 \end{bmatrix}$ $b = \begin{bmatrix} 15 \\ 82 \\ -5 \end{bmatrix}$

$U_1 = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 6 & 2 \\ 0 & 18 & 2 \end{bmatrix}$ Add $-4R_1 + R_2 : R_2$ Add $5R_1 + R_3 : R_3$ $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

$\boxed{U = U_2 = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix}}$ Add $-3R_2 + R_3 : R_3$ $\boxed{L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = L}$

$L = L_2$, $U = U_2$ $Lz = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 82 \\ -5 \end{bmatrix} = b$

$\Rightarrow z_1 = 15$, $4z_1 + z_2 = 82 \Rightarrow z_2 = 22$

$-5z_1 + 3z_2 + z_3 = -5 \Rightarrow z_3 = 4$ $\Big\}$ $z = \begin{bmatrix} 15 \\ 22 \\ 4 \end{bmatrix}$

$Ux = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 4 \end{bmatrix} = z \Rightarrow \begin{cases} -4x_3 = 4 \Rightarrow x_3 = -1 \\ 6x_2 + 2x_3 = 22 \Rightarrow x_2 = 4 \\ -3x_1 + x_2 - 2x_3 = 15 \Rightarrow x_3 = -3 \end{cases}$

$\boxed{\Rightarrow x = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}$

§ 1.8 1] Solve $L\underline{z} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underline{b} = \begin{bmatrix} 18 \\ 3 \\ 12 \end{bmatrix}$ by forward substitution:

$$z_1 = \frac{18}{2} = 9, \quad z_2 = \frac{3 - 2z_1}{-3} = \frac{-15}{-3} = 5$$

$$z_3 = \frac{12 + z_2 + z_1}{4} = \frac{8}{4} = 2$$

Solve $U\underline{x} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{z} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$ by back substitution:

$$x_3 = \frac{2}{2} = 1 \quad ; \quad x_2 = \frac{5 - x_3}{2} = \frac{4}{2} = 2$$

$$x_1 = \frac{9 - 0x_3 - 4x_2}{1} = \frac{1}{1} = 1.$$

Thus the solution is:

$$\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

§ 1.8 3] Solve $L\underline{z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \underline{b} = \begin{bmatrix} -2 \\ -2 \\ -16 \\ -66 \end{bmatrix}$ by forward substitution

$$z_1 = \frac{-2}{1} = 2 \quad ; \quad z_2 = \frac{-2 - 2z_1}{1} = \frac{2}{1} = 2$$

$$z_3 = \frac{-16 - 3z_2 + z_1}{1} = -24 \quad ; \quad z_4 = -\frac{66 - 2z_3 - 3z_2 - 4z_1}{1} = -16$$

Solve $U\underline{x} = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underline{z} = \begin{bmatrix} -2 \\ 2 \\ -24 \\ 16 \end{bmatrix}$ by back substitution.

$$x_4 = -\frac{16}{4} = -4 \quad ; \quad x_3 = \frac{-24 - 5x_4}{-2} = \frac{-4}{-2} = 2$$

$$x_2 = \frac{2 - x_4 - 3x_3}{-1} = \frac{0}{-1} = 0; \quad x_1 = \frac{-2 - x_4 + 0x_3 - 3x_2}{2} = \frac{2}{2} = 1$$

Thus the solution is:

$$\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -4 \end{bmatrix}$$

§ 1.8 $\boxed{8}$

$$A = \begin{bmatrix} -5 & 4 & 0 & 1 \\ -30 & 27 & 2 & 7 \\ 5 & 2 & 0 & 2 \\ 10 & 1 & -2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -17 \\ -102 \\ -7 \\ -6 \end{bmatrix} \quad A' = \begin{bmatrix} -5 & 4 & 0 & 1 \\ -30 & 27 & 2 & 7 \\ 10 & 1 & -2 & 1 \\ 5 & 2 & 0 & 2 \end{bmatrix}$$

0 diagonal entry is a problem

switch!

$$u_1 = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 9 & -2 & 3 \\ 0 & 6 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ \times -6 \\ \times 2 \\ \times 1 \end{array} \qquad L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \times -3 \\ \times -2 \end{array} \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

$$u = u_3 = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \times -\frac{1}{2} \end{array} \qquad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ -1 & 2 & 1/2 & 1 \end{bmatrix} = L$$

$Lu = A'$ $\longleftrightarrow$ NOTE $Lu \neq A$ but $P_{3,4} LU = A$ matrix that switches ROWS 3 & 4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} Lu = A \Rightarrow \underbrace{P(Lz) = b \Rightarrow (PL)z = b}$$

switches R_3 & R_4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -1 & 2 & 1/2 & 1 \\ -2 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -17 \\ -102 \\ -7 \\ -6 \end{bmatrix} \Rightarrow \left.\begin{array}{l} z_1 = -17 \\ z_2 = 0 \\ z_3 = -40 \\ z_4 = -4 \end{array}\right\} z = \begin{bmatrix} -17 \\ 0 \\ -40 \\ -4 \end{bmatrix}$$

$6z_1 + z_2 = -102 \Rightarrow z_2 = 0$

$-2z_1 + 3z_2 + z_3 = -6 \Rightarrow z_3 = -40$

$-z_1 + 2z_2 + \frac{1}{2} z_3 + z_4 = -7 \Rightarrow$

$$ux = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -17 \\ 0 \\ -40 \\ -4 \end{bmatrix} = z \Rightarrow$$

$x_4 = -4$

$-8x_3 = -40 \Rightarrow x_3 = 5$

$3x_2 + 2x_3 + x_4 = 0 \Rightarrow x_2 = -2$

$-5x_1 + 4x_2 + x_4 = -17 \Rightarrow x_1 = 1$

$$\Rightarrow \boxed{x = \begin{bmatrix} 1 \\ -2 \\ 5 \\ -4 \end{bmatrix}}$$