22A HW #5

§4.1 $\boxed{24}$

a) orthogonal (if $V_1 \cdot V_2 = 0$)

$(1,2)\cdot(-2,1) = -2+2=0$
$(-2,1)\cdot(2,4) = -4+4=0$
$(2,4)\cdot(-6,3) = -12+12=0$

$\boxed{\begin{matrix} u_1,u_4 & u_1,u_6 \\ u_4,u_5 & u_3,u_4 \\ u_5,u_6 & u_3,u_6 \end{matrix}}$

$(1,2)\cdot(-6,3) = -6+6=0$
$(-2,-4)\cdot(-2,1) = 4+-4=0$
$(-2,-4)\cdot(-6,3) = 12+-12=0$

b) same direction (if $V_1 = aV_2$, $a \in \mathbb{R}>0$)

$3(-2,1) = (-6,3)$ $\boxed{u_4,u_6}$
$2(1,2) = (2,4)$ $\boxed{u_1,u_5}$

c) opposite direction (if $V_1 = -aV_2$, $a \in \mathbb{R}>0$)

$-2(1,2) = (-2,-4)$ $\boxed{u_1,u_3}$
$-1(2,4) = (-2,-4)$ $\boxed{u_5,u_3}$

§4.1 $\boxed{26}$ $(a,2)\cdot(a,-2)=0$ if $(a,2)$ orthogonal to $(a,-2)$

$a^2-4=0$
$a^2=4$
$\Rightarrow \boxed{a=\pm 2}$

§4.1 $\boxed{T8}$ Show w orthogonal to u and v
$\Rightarrow w$ orthogonal to $ru+sv$, (r & s scalars)

w orthogonal to $u,v \Rightarrow w\cdot u=0$ and $w\cdot v=0$

Consider $w\cdot(ru+sv)$
$= w\cdot ru + w\cdot sv$
$= r(w\cdot u) + s(w\cdot v)$
$= r(0)+s(0) = 0+0 = 0$

$\Rightarrow w$ orthogonal to $ru+sv$, as required ∎

§4.2 $\boxed{14}$ Need $a,b,c \in \mathbb{R}$ such that
$a(1,2,-3)+b(-1,1,1)+c(-1,4,-1)=(2,-2,3)$

$$\left.\begin{aligned} a-b-c &= 2 \\ 2a+2b+4c &= -2 \\ -3a+b-c &= 3 \end{aligned}\right\} \quad \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 4 \\ -3 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 2 & 1 & 4 & -2 \\ -3 & 1 & -1 & 3 \end{array}\right] \xrightarrow[3R_1+R_3:R_3]{-2R_1+R_2:R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 6 & -6 \\ 0 & -2 & -4 & 9 \end{array}\right]$$

$$\xrightarrow[R_3\div -2]{R_2\div 3} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & -\frac{9}{2} \end{array}\right] \xrightarrow[-R_2+R_3:R_3]{} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -\frac{1}{2} \end{array}\right] \Longleftrightarrow \text{Impossible}$$

$\boxed{\Rightarrow (2,-2,3) \text{ is not a linear combination of these 3 vectors}}$

§4.2 $\boxed{30}$

a) $2i+3j-4k$ $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$

b) $i+2j$ $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

c) $-3i$ $\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$

d) $3i-2k$ $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$

§4.2 $\boxed{T10}$ Show $\|u+v\|^2 = \|u\|^2 + \|v\|^2 \Longleftrightarrow u\cdot v = 0$

$$\begin{aligned} \|u+v\|^2 &= (u+v)\cdot(u+v) \\ &= u\cdot u + 2(u\cdot v) + v\cdot v \\ &= \|u\|^2 + 2(u\cdot v) + \|v\|^2 \end{aligned}$$

$\Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$ if & only if $u\cdot v = 0$, as req'd ■

§4.2 $\boxed{T13}$ $\|u+v\|^2 + \|u-v\|^2$

$$\begin{aligned} &= (u+v)\cdot(u+v) + (u-v)\cdot(u-v) \\ &= u\cdot u + 2(u\cdot v) + v\cdot v + u\cdot u - 2(u\cdot v) + v\cdot v \\ &= \|u\|^2 + 2(u\cdot v) + \|v\|^2 + \|u\|^2 - 2(u\cdot v) + \|v\|^2 \\ &= 2\|u\| + 2\|v\| \end{aligned}$$

as req'd ■

§ 4.2 $\boxed{T15}$ $\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$

$= \frac{1}{4}\left(\|u\|^2 + \|v\|^2 + 2(u\cdot v) - \left[\|u\|^2 + \|v\|^2 - 2(u\cdot v)\right]\right)$

$= \frac{1}{4}\left[4(u\cdot v)\right]$

$= u\cdot v$

§ 4.3 $\boxed{4}$ NTS $L(u+v) = L(u) + L(v)$ and $L(cu) = cL(u)$ $\forall\ u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$

ⓐ $$L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \\ u_4+v_4 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1 \\ (u_1+v_1)^2 + (u_2+v_2) \\ (u_1+v_1) - (u_3+v_3) \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ (u_1^2 + v_1^2 + 2u_1v_1) + (u_2+v_2) \\ (u_1+v_1) - (u_3+v_3) \end{bmatrix}$$

$$L(u) + L(v) = \begin{bmatrix} u_1 \\ u_1^2 + u_2 \\ u_1 - u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_1^2 + v_2 \\ v_1 - v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ (u_1^2 + v_1^2) + (u_2+v_2) \\ (u_1+v_1) - (u_3+v_3) \end{bmatrix}$$

$L(u+v) \neq L(u) + L(v) \Rightarrow L$ NOT a linear trans.

ⓑ $$L(u+v) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = L(u) + L(v)$$

$$L(cu) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = c\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = cL(u)$$

$\Rightarrow L$ IS a linear transformation.

ⓒ $$L(u+v) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = L(u) + L(v)$$

$$L(cu) = L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = cL(u)$$

$\Rightarrow L$ IS a linear transformation

b, c are linear transformation
a IS NOT

§4.3 $\boxed{14}$ $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is w in range L?

(a) $w = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ NTS $\exists \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ st $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = w$

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \left.\right\} \begin{aligned} -x + 2y \quad &= 1 \\ x + y + z &= 2 \\ 2x - y + z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & -1 \end{array}\right] \xrightarrow[\substack{2R_1+R_3 : R_3 \\ -1R_1 : R_1}]{R_1+R_2 : R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 0 & 3 & 1 & 1 \end{array}\right] \xrightarrow[: R_3]{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right] \leftarrow \text{impossible}$$

$\boxed{\text{Inconsistent system} \Rightarrow w \text{ \underline{NOT} in range of } L}$

(b) $w = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & -1 & 1 & 2 \end{array}\right] \xrightarrow[\text{as above}]{\text{same row ops}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 3 & 1 & 4 \\ 0 & 3 & 1 & 4 \end{array}\right] \xrightarrow[\text{op}]{\text{same}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{\text{Consistent system} \Rightarrow w \text{ \underline{IS} in the range of } L}$

§4.3 $\boxed{18}$ $L: \mathbb{R}^3 \to \mathbb{R}^3$ st $L(i) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $L(j) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $L(k) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Find $L\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = L\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) + L\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}\right) + L\left(\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right)$

$$= 2L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + (-1)L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + 3L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= 2L(i) - L(j) + 3L(k)$$

$$= 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 1 + 3 \\ 4 - 0 + 3 \\ -2 - 2 + 9 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix} = L\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right)}$$

§ 4,3 $\boxed{26}$ $L: \mathbb{R}^2 \to \mathbb{R}^2$ find Standard matrix A representing L

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = col_1(A)$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0-1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = col_2(A)$$

$$\Rightarrow \boxed{A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}$$

§ 4,3 $\boxed{28}$ $L: \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise rotation thru $\frac{\pi}{4}$ radians
Find St. Matrix A representing L

$$L(u) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} u$$ (gives counter clockwise rotation through ϕ)

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$ for $\phi = \pi/3$, $u = \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{bmatrix}$$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = col_1(A)$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} = col_2(A)$$

$$\Rightarrow \boxed{A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}}$$

§ 4.3 $\boxed{T9}$ $L: \mathbb{R}^2 \to \mathbb{R}^2$ st $L(u) = Au$ with $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$
(counterclockwise rotation through ϕ)

Let $\phi = 30°$

See next pg for matrix calculations

ⓐ if $T_1(u) = A^2(u)$ $\boxed{T_1 \text{ defines counter clockwise rotation by } 60° \text{ (apply A twice)}}$

ⓑ if $T_2(u) = A^{-1}(u)$ $\boxed{T_2 \text{ defines clockwise rotation by } 30°}$

ⓒ if $T(u) = A^k u = u$ $\boxed{\text{The smallest positive value for } K \text{ is } 12 \text{ if } T(u) = A^k u = u}$

$30° \cdot 12 = 360°$

$\Rightarrow$ apply A 12 times to get back to original u

continued
§ 4.3
T9) (a)

$$A^2 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\phi - \sin^2\phi & -2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & \cos^2\phi - \sin^2\phi \end{bmatrix}$$

need trig identities!

$$= \begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \quad \text{for } \phi = 30^\circ$$

$\Rightarrow$ counter clockwise rotation by 60°

(b) $$A^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{-1}$$

$$= \frac{1}{\cos^2\phi + \sin^2\phi}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

another trig ID

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{bmatrix}$$

think about unit circle

$$= \begin{bmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix}$$

$\Rightarrow$ clockwise rotation by 30°

(c) using part (a);

$A^k u$ rotates u $(30k)^\circ$ counter clockwise

Need to find k s.t. $(30k)^\circ = 360^\circ$ or multiple of 360°

$\Rightarrow$ smallest $k > 0$ to satisfy this relationship is $k = 12$.

Note that any multiple of $k = 12$ will also satisfy $A^k u = u$

HW # 6 , 22A

§ 6.1 $\boxed{4}$ $V = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = d, \oplus = \text{matrix add}, \odot = \text{scalar mult} \}$

$\boxed{\text{closed under } \oplus}$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$\vec{v}_1, \vec{v}_2 \in V \Rightarrow a_1 = d_1$ and $a_2 = d_2$

$\Rightarrow a_1 + a_2 = d_1 + d_2 \Rightarrow \vec{v}_1 + \vec{v}_2 \in V$

$\boxed{\text{closed under } \odot}$

$s \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, s a scalar

$v_1 \in V \Rightarrow a = d \Rightarrow sa = sd \Rightarrow sv_1 \in V$

§ 6.1 $\boxed{12}$ $\{ x, y, z : (x,y,z) \oplus (x', y', z') = (x+x', y+y', z+z')$ and $c \odot (x, y, z) = (x, 1, z) \}$

$\boxed{\text{IS \underline{NOT} a vector space. Properties e, f, \& h (as listed on pg 272) fail}}$

e) $c \odot (u+v) = c \odot (u_1+v_1, u_2+v_2, u_3+v_3) = (u_1+v_1, 1, u_3+v_3)$ $\neq$
$c \odot u \oplus c \odot v = (u_1, 1, u_3) \oplus (v_1, 1, v_3) = (u_1+v_1, 2, u_3+v_3)$

f) $(c+d) \odot u = (u_1, 1, u_3) \neq$
$c \odot u \oplus d \odot u = (u_1, 1, u_3) \oplus (u_1, 1, u_3) = (2u_1, 2, 2u_3)$

h) $1 \odot u = (u_1, 1, u_3) \neq u = (u_1, u_2, u_3)$ unless $u_2 = 1$,
$\therefore$ not true for all $u \in V$

§ 6.1 $\boxed{T4}$ Show $au = bu$ and $u \neq 0 \Rightarrow a = b$

$au = bu \Rightarrow au - bu = 0$
$\Rightarrow (a-b)u = 0$
$\Rightarrow$ either $a - b = 0$ or $u = 0$ (Thm 6.1 c)
$\Rightarrow a - b = 0$ since $u \neq 0$
$\Rightarrow a = b$, as required ∎

§6.1 T5] Show a vector space has only one zero vector.

Pf Let $u \in V$ be a zero vector in V.

Suppose $v \in V$ is also a zero vector, but $u \neq v$.

Then $u \oplus v = u$ and $u \oplus v = v$.

Therefore $u = v$.

§6.1 T6] Show a vector $u \in V$ has only one negative.

Let $u \in V$ and $(-u)$ be a negative of u.

Suppose v is also a negative of u, but $(-u) \neq v$.

Then $u \oplus (-u) = 0$ and $u \oplus v = 0$.

So: $u \oplus (u \oplus (-u)) = u \oplus (u \oplus v)$

$\Rightarrow (u \oplus u) \oplus (-u) = (u \oplus u) \oplus v$

$\Rightarrow (-u) = v$